On Some Properties of Intuitionistic Fuzzy Implications

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Abstract

We discuss the algebraic properties of intuitionistic fuzzy implications. We examine the conjugacy problem in this family of functions. The characterizations of intuitionistic fuzzy $\mathcal{S}$-implications and some lattice properties of intuitionistic fuzzy implications are presented.

Keywords: Intuitionistic fuzzy implication, $\mathcal{S}$-implication, conjugate implications.

1 Introduction

Intuitionistic fuzzy sets were introduced by Atanassov in 1983 in the following way.

Definition 1 ([1]). An intuitionistic fuzzy set A in a universe X is an object

$$A = \{(x, \mu_A(x), \nu_A(x)) : x \in X\},$$

where functions $\mu_A \colon X \to [0,1]$, $\nu_A \colon X \to [0,1]$ are called, respectively, the membership degree and the non-membership degree. They satisfy the condition $\mu_A(x) + \nu_A(x) \leq 1$ for all $x \in X$.

This family can be seen as L-fuzzy set in the sense of Goguen. We use in this paper the following notation presented by Cornelis et al. [7]:

$$L = \{(x_1, x_2) \in [0,1]^2 : x_1 + x_2 \leq 1\},$$

$$(x_1, x_2) \leq_L (y_1, y_2) \iff x_1 \leq y_1 \wedge x_2 \geq y_2, \quad (x_1, x_2), (y_1, y_2) \in L.$$

It can be easily proved that $(L, \leq_L)$ is a complete lattice with units $0_l = (0,1)$ and $1_L = (1,0)$. This lattice is not linear.

Like in the fuzzy set theory we can consider the generalizations of classical logical connectives to the lattice L. In last years many papers are dedicated to investigations of these operations (see e.g. [4], [7], [8] or [10]). In this talk we want to focus on the properties of intuitionistic fuzzy implications. Our main goal is to present different theorems related to the problem of the characterization of intuitionistic fuzzy implications and to examine the problem of the conjugacy.

2 Basic definitions

First we remind basic notations that will be useful in the sequel.

Definition 2. A function $\mathcal{N} \colon L \to L$ is called an intuitionistic fuzzy negation (shortly *IF* negation) if it is decreasing and satisfies $\mathcal{N}(0_L) = 1_L$, $\mathcal{N}(1_L) = 0_L$. If, in addition, $\mathcal{N}$ is an involution,

$$\mathcal{N}(\mathcal{N}(x)) = x, \qquad x \in L,$$

then $\mathcal{N}$ is called a strong *IF* negation.

The characterization of strong *IF* negations was first investigated by Bustince et al. [4]. The next result was obtained by Deschrijver et al.

Theorem 1 ([9]). *A function $\mathcal{N} \colon L \to L$ is a strong IF negation if, and only if, there exists a strong negation $N \colon [0,1] \to [0,1]$ such that*

$$\mathcal{N}(x) = (N(1 - x_2), 1 - N(x_1)), \quad x \in L.$$

The definition of intuitionistic fuzzy t-norms and t-conorms are similar to the classical case.

Definition 3. A function $\mathcal{T}\colon L^2 \to L$ is called an intuitionistic fuzzy triangular norm (shortly IF t-norm) if it is commutative, associative and increasing operation with the neutral element 1_L. A function $\mathcal{S}\colon L^2 \to L$ is called an intuitionistic fuzzy triangular conorm (shortly IF t-conorm) if it is commutative, associative and increasing operation with the neutral element 0_L.

Lemma 1 (cf. [7]). *If T is a t-norm and S is a t-conorm such that $T(x,y) \leq 1 - S(1-x, 1-y)$ then functions $\mathcal{T}, \mathcal{S}\colon L^2 \to L$ defined by*

$$\mathcal{T}((x_1,x_2),(y_1,y_2)) = (T(x_1,y_1), S(x_2,y_2)),$$
$$\mathcal{S}((x_1,x_2),(y_1,y_2)) = (S(x_1,y_1), T(x_2,y_2)),$$

for all $(x_1,x_2),(y_1,y_2) \in L$ are an IF t-norm and IF t-conorm, respectively. In both case we say that $\mathcal{T}$ and $\mathcal{S}$ are t-representable.

It is interesting and important that not every IF t-norm and IF t-conorm have these representations.

The definition of the intuitionistic implication is based on the notation from fuzzy set theory introduced by Fodor, Roubens [11].

Definition 4 (see [6]). A function $\mathcal{I}\colon L^2 \to L$ is called an intuitionistic fuzzy implication (shortly IF implication) if it is monotonic with respect to both variables (separately) and fulfills the border conditions $\mathcal{I}(0_L, 0_L) = \mathcal{I}(0_L, 1_L) = \mathcal{I}(1_L, 1_L) = 1_L$, $\mathcal{I}(1_L, 0_L) = 0_L$. The set of all intuitionistic fuzzy implications is denoted by IFI.

3 Classes of IF implications

Two important classes of IFI, which are the generalizations from the fuzzy logic, are investigated in the literature.

Definition 5 ([7]). Let $\mathcal{S}\colon L^2 \to L$ be an IF t-conorm and $\mathcal{N}\colon L \to L$ be an IF negation. A function $\mathcal{I}_{\mathcal{S},\mathcal{N}}\colon L^2 \to L$ defined by formula

$$\mathcal{I}_{\mathcal{S},\mathcal{N}}(x,y) = \mathcal{S}(\mathcal{N}(x), y), \qquad x, y \in L$$

is called an IF $\mathcal{S}$-implication. If $\mathcal{S}$ is t-representable then $\mathcal{I}_{\mathcal{S},\mathcal{N}}$ is called a t-representable IF $\mathcal{S}$-implication.

The properties and characterizations of this subclasses of IFI were investigated by Cornelis et al. [7] and Bustince et al. [5].

Theorem 2 ([5]). *An intuitionistic fuzzy implication $\mathcal{I}$ is an* IF *$\mathcal{S}$-implication based on the strong IF negation $\mathcal{N}(x_1,x_2) = (x_2,x_1)$ and on the t-representable t-conorm $\mathcal{S}$ (for which fuzzy T and S are dual) if, and only if, $\mathcal{I}$ satisfies*

$$\mathcal{I}(1,x) = x, ,$$
$$\mathcal{I}(x, \mathcal{I}(y,z)) = \mathcal{I}(y, \mathcal{I}(x,z)),$$
$$\mathcal{I}((x_1,x_2),(y_1,y_2)) = \mathcal{I}((y_2,y_1),(x_2,x_1)),$$

and there exists a fuzzy implication $I\colon [0,1]^2 \to [0,1]$ such that

$$\mathcal{I}((x_1,x_2),(y_1,y_2)) = (I(1-x_2, y_1), 1 - I(x_1, 1-y_2)).$$

We investigated deeper above theorem and as a result we obtain the following.

Proposition 1. *An intuitionistic fuzzy implication $\mathcal{I}$ is an IF $\mathcal{S}$-implication based on the strong IF negation $\mathcal{N}(x_1,x_2) = (x_2,x_1)$ and on the t-representable t-conorm $\mathcal{S}$ (for which fuzzy T and S are dual) if, and only if, there exists a fuzzy S-implication $I\colon [0,1]^2 \to [0,1]$ based on the Łukasiewicz negation $N(x) = 1-x$ such that*

$$\mathcal{I}((x_1,x_2),(y_1,y_2)) = (I(1-x_2, y_1), 1 - I(x_1, 1-y_2)).$$

Theorem 3. *A function $\mathcal{I}\colon L^2 \to L$ is an IF $\mathcal{S}$-implication based on a strong IF negation $\mathcal{N}$ if, and only if, $\mathcal{I} \in IFI$ satisfies conditions*

$$\mathcal{I}(1,x) = x, \qquad x \in L,$$
$$\mathcal{I}(x, \mathcal{I}(y,z)) = \mathcal{I}(y, \mathcal{I}(x,z)), \qquad z, y, z \in L,$$
$$\mathcal{I}(\mathcal{I}(x,0),0) = x, \qquad x \in L.$$

Theorem 4. *A function $\mathcal{I}\colon L^2 \to L$ is an t-representable IF $\mathcal{S}$-implication based on some strong IF negation $\mathcal{N}$ if, and only if, there exist fuzzy S-implications $I, J\colon [0,1]^2 \to [0,1]$ and a strong negation $N\colon [0,1] \to [0,1]$ such that*

$$\mathcal{I}((x_1,x_2),(y_1,y_2)) = (I(N(x_2), y_1), N(J(x_1, N(y_2)))).$$

We can see, that if $I = J$ and N is the Łukasiewicz negation than we have the formula obtained by Butinice et al. in Theorem 2.

Definition 6 ([7]). Let $\mathcal{T}\colon L^2 \to L$ be an *IF* t-norm which satisfies

$$\sup_{z\in Z} \mathcal{T}(x,z) = \mathcal{T}(x, \sup_{z\in Z} z), \quad x \in L, \ \ Z \subset L.$$

A function $\mathcal{I}_{\mathcal{T}}\colon L^2 \to L$ defined by formula

$$\mathcal{I}_{\mathcal{T}}(x,y) = \sup\{t \in L : \mathcal{T}(x,t) \le y\}, \quad x, y \in L$$

we call an *IF* $\mathcal{R}$-implication.

4 Problem of the Conjugacy

Since many theorems connected with the characterizations of fuzzy operators use the increasing bijections, we state now the important result, which shows the dependence between increasing bijections on L and on the unit interval.

Theorem 5 ([7]). *A function $\Phi\colon L \to L$ is an continuous increasing bijection if, and only if, there exists a continuous increasing bijection $\varphi\colon [0,1] \to [0,1]$ such that*

$$\Phi(x) = (\varphi(x_1), 1 - \varphi(1 - x_2)), \quad x \in L.$$

We say that the functions $F, G\colon L^2 \to L$ are conjugate, if there exists a continuous increasing bijection $\Phi\colon L \to L$ such that $G = F_\varphi$, where

$$F_\Phi(x,y) = \Phi^{-1}(F(\Phi(x), \Phi(y))), \quad x, y \in [0,1].$$

The problem of the conjugacy in the family of fuzzy implications was investigated in [2]. Here we present interesting facts concerning intuitionistic operators.

Proposition 2. *Let $\Phi\colon L \to L$ be a continuous increasing bijection. If $\mathcal{T}$ ($\mathcal{S}$) is an IF t-norm (t-conorm), then the function $\mathcal{T}_\Phi$ ($\mathcal{S}_\Phi$) is also an IF t-norm (t-conorm).*

Theorem 6. *Let $\Phi\colon L \to L$ be a continuous increasing bijection. If $\mathcal{I}$ is an IF $\mathcal{S}$-implication based on some IF t-conorm $\mathcal{S}$ and strong IF negation $\mathcal{N}$, then the function $\mathcal{I}_\Phi$ is also an IF $\mathcal{S}$-implication based on the IF t-conorm $\mathcal{S}_\Phi$ and the strong IF negation $\mathcal{N}_\Phi$.*

Theorem 7. *Let $\Phi\colon L \to L$ be a continuous increasing bijection. If $\mathcal{I}$ is an IF R-implication based on some IF t-norm $\mathcal{T}$, then the function $\mathcal{I}_\Phi$ is also an R-implication based on the t-norm $\mathcal{T}_\Phi$.*

5 Lattice of IF implications

Now we show, that the set IFI has analogous lattice properties like the family of all fuzzy implications (see [3]).

Theorem 8. *The family $(IFI, \inf, \sup)$ is a complete lattice, i.e.*

$$\forall_{t\in T} \ \ (\mathcal{I}_t \in IFI) \Longrightarrow \sup_{t\in T} \mathcal{I}_t, \inf_{t\in T} \mathcal{I}_t \in IFI.$$

Corollary 1. *IFI has the greatest element*

$$\mathcal{I}_1(x,y) = \begin{cases} 1_L, & \text{if } x < 1_L \text{ or } y > 0_L \\ 0_L, & \text{if } x = 1_L \text{ and } y = 0_L \end{cases},$$

and the least element

$$\mathcal{I}_0(x,y) = \begin{cases} 1_L, & \text{if } x = 0_L \text{ or } y = 1_L \\ 0_L, & \text{if } x > 0_L \text{ and } y < 1_L \end{cases}$$

for $x, y \in L$.

Definition 7. Set X of linear space is convex over $\mathbb{R}$ if, with any two points, $x, y \in X$, X contains a line segment between x and y, i.e.

$$\forall_{\lambda\in[0,1]} \ z = \lambda x + (1-\lambda) y \in X.$$

Theorem 9. *IFI is a convex set of functions.*

The above theorem brings a tool for generation of parameterized families of fuzzy implications.

At the end we consider the law of contraposition and the conjugation property in the lattice IFI.

Proposition 3. *Let $\mathcal{I}, \mathcal{J} \in IFI$, $\mathcal{N}$ be a strong IF negation. The operation defined by formula*

$$\mathcal{I}_{\mathcal{N}}(x,y) = \mathcal{I}(\mathcal{N}(y), \mathcal{N}(x)) \quad x, y \in L.$$

is order preserving (isotone), i.e.

$$\mathcal{I} \le \mathcal{J} \Rightarrow \mathcal{I}_{\mathcal{N}} \le \mathcal{J}_{\mathcal{N}}.$$

Moreover, for $\mathcal{I}_t \in IFI$, $t \in T$, we get

$$(\sup_{t\in T} \mathcal{I}_t)_{\mathcal{N}} = \sup_{t\in T} (\mathcal{I}_t)_{\mathcal{N}}, \quad (\inf_{t\in T} \mathcal{I}_t)_{\mathcal{N}} = \inf_{t\in T} (\mathcal{I}_t)_{\mathcal{N}}.$$

Proposition 4. *For any $\mathcal{I}, \mathcal{J} \in IFI$, and continuous increasing bijection $\Phi\colon L \to L$ we have*

$$I \le J \Longleftrightarrow I_\Phi \le J_\Phi.$$

Moreover, for $\mathcal{I}_t \in IFI$, $t \in T$, we get

$$(\sup_{t\in T} \mathcal{I}_t)_\Phi = \sup_{t\in T} (\mathcal{I}_t)_\Phi, \quad (\inf_{t\in T} \mathcal{I}_t)_\Phi = \inf_{t\in T} (\mathcal{I}_t)_\Phi.$$

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