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Conditional probability on the Kôpka D-posets

Ivana Hollá and Karol Samuelčík

DEPARTMENT OF MATHEMATICS, FACULTY OF NATURAL SCIENCES,

MATHIAE BELII UNIVERSITY

Tajovského 40, 974 01 Banská Bystrica, Slovakia

e-mail: holla@fpv.umb.sk,

karolsamuelcik@centrum.sk

Abstract

The conditional probability on the Kôpka D-poset is studied. The notions Kôpka D-poset and conditional probability is introduced. The basic properties of conditional probability on the Kôpka D-posets is proved.
Keywords: Kôpka D-posets, conditional propability

1 Introduction

In quantum structures play an important function the notion D-poset. It was standing independently in Slovak school (D-posets,[9]) and American school (effect algebras,[5]). D-posets with the new property product create a good space for the hopefull probability aplications.

In this paper will by the main idea similarly as in [10]and [17] builted on distribution function only. The existence of joint distribution and convergence theorem has been proved in [10], the Central limit theorem has been proved in [17]. We lead notion conditional probability on the Kôpka D-posets in the initiated meaning.

2 Kôpka D-poset -basic notions

Definition 1 *By a D-poset D it is consider the algebraic structure* $D = (D, \leq, -, 0, 1)$ *such that:*

1) $\leq$ *is a partial ordering on* D *with the least element* 0 *and the greatest element* 1.

2) $- : D \times D \to D$ *is a partial binary operation, where* $b - a$ *is defined iff* $a \leq b$ *and*

- *i)* $b - a \leq b$,
- *ii)* $b - (b - a) = a$,
- *iii)* $a \leq b \leq c \Rightarrow c - b \leq c - a,\ (c - a) - (c - b) = b - a$.

Definition 2 *Let system* $D = (D, \leq, -, 0, 1)$ *be a D-poset. It is called the Kôpka D-poset, if there is a binary operation* $* : D \times D \longrightarrow D$*, which is commutative, associative, and*

i) $a * 1 = a, a \in D$,
ii) $a \leq b \Rightarrow a * c \leq b * c,\ a, b, c \in D$,
iii) $a - (a * b) \leq 1 - b,\ a, b, c \in D$.

Example 1 *Let* (Ω, S, P) *be a classical Kolmogorov probability space,* F *be a system of fuzzy events,*

$$F = \{\mu_A : \Omega \longrightarrow \langle 0, 1\rangle, \mu_A \text{ is } S\text{-measurable}\}.$$

Partial ordering is defined

$$\mu_A \leq \mu_B \Longleftrightarrow \mu_A(\omega) \leq \mu_B(\omega),\ \forall \omega \in \Omega$$

With respect to this ordering the least element is 0_Ω *and the greatest element is constant function* 1_Ω.
Binary operations $" - ", " * "$ *can be defined:*

$- : F \times F \longrightarrow F$, *iff* $\mu_A \leq \mu_B$ *then* $(\mu_B - \mu_A)(\omega) = \mu_B(\omega) - \mu_A(\omega),\ \forall \omega \in \Omega$,

$* : F \times F \longrightarrow F$, *iff* $\mu_A * \mu_B(\omega) = \mu_A(\omega).\mu_B(\omega),\ \forall \omega \in \Omega$.

The algebraic structure $F = (F, \leq, -, *, 0_\Omega, 1_\Omega)$ *is Kôpka D-poset.*

Definition 3 *Kôpka D-poset* D *with the following property:*

$$\text{if } k \leq l \text{ then } a * (l - k) = a * l - a * k,\ k, l, a \in D,$$

is called strong Kôpka D-poset.

Definition 4 *A state on a Kôpka D-poset* D *is any mapping* $m : D \to \langle 0, 1\rangle$ *satisfying the following properties:*

- *i)* $m(1) = 1, m(0) = 0$,
- *ii)* $a_n \nearrow a \Rightarrow m(a_n) \nearrow m(a)$,
- *iii)* $a_n \searrow a \Rightarrow m(a_n) \searrow m(a)$.

Definition 5 *D-additive state on a Kôpka D-poset D is any mapping $m : D \to \langle 0, 1\rangle$, where*

$$a \leq b \Longrightarrow m(b) = m(b - a) + m(a)$$

Definition 6 *Let $J = \{(-\infty, t); t \in R\}$. An observable on D is any mapping $x : J \to D$ satisfying the following conditions:*

- *i)* $A_n \nearrow R \Rightarrow x(A_n) \nearrow 1$,
- *ii)* $A_n \searrow \emptyset \Rightarrow x(A_n) \searrow 0$,
- *iii)* $A_n \nearrow A \Rightarrow x(A_n) \nearrow x(A)$.

3 Conditional probability

Conditional probability (of A with respect to B) is the real number $P(A|B)$ such that

$$P(A \cap B) = P(B).P(A|B).$$

When A, B are independent then $P(A|B) = P(A)$, then event A does not depend on the occuring of event B. Another point of view:

$$P(A \cap B) = \int_B P(A|B)dP.$$

The number $P(A|B)$ can be regarded as a constant function. Constant functions are measurable with respect to the σ-algebra $S_0 = \{\emptyset, \Omega\}$,

$$\{\omega \in \Omega; f(\omega) \in C\} = \emptyset \text{ or } \{\omega \in \Omega; f(\omega) \in C\} = \Omega.$$

Generally $P(A|S_0)$ can be defined for any σ-algebra $S_0 \subset S$, as an S_0-measurable function such that

$$P(A \cap C) = \int_C P(A|S_0)dP, C \in S_0.$$

If $S_0 = S$, then we can put $P(A|S_0) = \chi_A$ since χ_A is S_0-measurable, and

$$\int_C \chi_A dP = \int_\Omega \chi_C \chi_A dP = \int_\Omega \chi_{A\cap C} dP = P(A \cap C).$$

An importat example of S_0 is the family of all pre-images of a random variable $\xi : \Omega \to R$

$$S_0 = \{\xi^{-1}(B); B \in \sigma(J)\}$$

In this case we shall write $P(A|S_0) = P(A|\xi)$, hence

$$\int_C P(A|\xi)dP = P(A \cap C), C = \xi^{-1}(B), B \in \sigma(J).$$

By the transformation formula

$$P(A \cap \xi^{-1}(B)) = \int_{\xi^{-1}(B)} g \circ \xi dP = \int_B g dP_\xi, B \in \sigma(J).$$

Proposition 1 *Let D be a Kôpka D-poset, $m : D \to \langle 0,1\rangle$ be a state, $x : J \to D$ be an observable. Define $F : R \to \langle 0,1\rangle$ by the formula*

$$F(t) = m(x((-\infty, t))).$$

Then F has the following properties:

- *(i) F is non-decreasing,*
- *(ii) F is left continuos in any point $t_0 \in R$,*
- *(iii) $lim_{t\to\infty} F(t) = 1$,*
- *(iv) $lim_{t\to-\infty} F(t) = 0$.*

Proof. Let $t < s$, put $t_1 = t$, $t_n = s$ $(n = 1, 2, 3, ...)$. Then $t_n \nearrow s$, hence

$$F(t_n) = m(x((-\infty, t_n))) \nearrow m(x((-\infty, s))) = F(s).$$

Therefore $F(t) = F(t_1) \leq F(s)$, hence F is non-decreasing. If $t_n \nearrow t$, then $x((-\infty, t_n)) \nearrow x((-\infty, t))$, hence

$$F(t_n) = m(x((-\infty, t_n))) \nearrow m(x((-\infty, t))) = F(t),$$

and therefore F is left continuous in t. Similarly the equalities $F(\infty) = 1$, $F(-\infty) = 0$ can be proved. ∎

Remark 1 *There exists exactly one measure*

$$\lambda_F : B(R) \to \langle 0,1\rangle,$$

such that

$$\lambda_F(\langle u, v)) = F(v) - F(u),$$

and there holds the following equalities

$$\lambda_F(\langle u, v)) = F(v) - F(u) = m(x((-\infty, v))) - m(x((-\infty, u))) =$$
$$m(x((-\infty, v)) - x((-\infty, u))).$$

Proposition 2 *Let D be a Kôpka D-poset, $a \in D$, $m : D \to \langle 0,1\rangle$ be a state, $x : J \to D$ be an observable. Define $G : R \to \langle 0,1\rangle$ by the formula*

$$G(t) = m(a * x((-\infty, t)))$$

Then G has the following properties:

- *(i) G is non-decreasing,*
- *(ii) G is left continuos in any point $t_1 \in R$,*
- *(iii) $lim_{t\to\infty} G(t) = 1$,*
- *(iv) $lim_{t\to-\infty} G(t) = 0$.*

Proof. Let $G(t) = m(a * x((-\infty, t)))$ and $G : R \to \langle 0,1\rangle$.

- (i) G is non-decreasing
  Let $t_1 < t_2$.
  Then

$$x((-\infty, t_1)) \leq x((-\infty, t_2)) \Longrightarrow a * x((-\infty, t_1)) \leq a * x((-\infty, t_2))$$

  Hence

$$G(t_1) = m(a * x((-\infty, t_1))) \leq m(a * x((-\infty, t_2))) = G(t_2)$$

- (ii) G is left continuos in any point $t_1 \in R$.
  Let $t_n \nearrow t_1$.
  Then

$$x((-\infty, t_n)) \nearrow x((-\infty, t_1)) \Longrightarrow a * x((-\infty, t_n)) \nearrow a * x((-\infty, t_1))$$

  Hence

$$G(t_n) = m(a * x((-\infty, t_n))) \nearrow m(a * x((-\infty, t_1))) = G(t_1)$$

- (iii) $lim_{t\to\infty} G(t) = 1$.
  Let $t_n \nearrow \infty$.
  Then

$$lim_{t\to\infty} G(t_n) = m(a * x((-\infty, \infty)) = m(a * 1) = 1$$

- (iv) $lim_{t\to-\infty} G(t) = 0$.
  Let $t_n \searrow -\infty$.
  Then

$$lim_{t\to-\infty}G(t_n) = m(a * 0) = 0.$$

Remark 2 *The function $G : R \to \langle 0,1\rangle$ defined by* ∎

$$G(t) = m(a * x((-\infty, t)))$$

is a distribution function of observable x. There exists exactly one measure

$$\lambda_G : B(R) \to \langle 0,1\rangle,$$

such that

$$\lambda_G(\langle u, v)) = G(u) - G(v).$$

Proposition 3 *Let D be a Kôpka D-poset, $a \in D$, $m : D \longrightarrow \langle 0,1\rangle$ be a D-additive state, $x : J \longrightarrow D$ be an observable, $F, G : R \longrightarrow \langle 0,1\rangle$ be distribution functions*

$$F(t) = m(x((-\infty, t)))$$

and

$$G(t) = m(a * x((-\infty, t))).$$

Then for the Lebesque-Stielties measures λ_G, λ_F there holds following

$$\lambda_G \leq \lambda_F.$$

Proof.

$$\lambda_G(\langle u, v)) = G(u) - G(v) = m(a * x((-\infty, v))) - m(a * x((-\infty, u))) =$$

$$= m(a * (x((-\infty, v)) - x((-\infty, u)))) \leq m(x((-\infty, v)) - x((-\infty, u))) = \lambda_F(\langle u, v)).$$ ∎

Theorem 1 *Let D be a Kôpka D-poset, $a \in D$, $m : D \longrightarrow \langle 0,1\rangle$ be a D-additive state, $x : J \longrightarrow D$ be an observable, $F, G : R \longrightarrow \langle 0,1\rangle$ be distribution functions*

$$F(t) = m(x((-\infty, t)))$$

and

$$G(t) = m(a * x((-\infty, t))).$$

Then there exists function $f : R \longrightarrow R$, such that

$$\int_{(-\infty,t)} f d\lambda_F = m(a * x((-\infty, t))).$$

Proof. We know, that

$$\lambda_G(\langle u, v)) \leq \lambda_F(\langle u, v)).$$

for every $u, v \in R$, $u \leq v$. Therefore $\lambda_G(B) \leq \lambda_F(B)$ for any $B \in \sigma(J)$. Assume, that $\lambda_F(B) = 0$, then there holds

$$0 \leq m(a * x(B)) = \lambda_G(B) \leq \lambda_F(B) = 0,$$

and according to Radon-Nikodym theorem there exists function f such that

$$\lambda_G(B) = \int_B f d\lambda_F,$$

for every $B \in \sigma(J)$. ∎

Definition 7 *Let D be a Kôpka D-poset, $a \in D$, $m : D \longrightarrow \langle 0, 1\rangle$ be a D-additive state, $x : J \longrightarrow D$ be an observable. Then the conditional probability $p(a|x) : R \longrightarrow R$, is a Borel measurable function (i.e. $B \in J \Longrightarrow f^{-1}(B) \in \sigma(J)$ such that*

$$\int_{(-\infty,t)} f d\lambda_F = m(a * x(-\infty, t)),$$

for any $t \in R$.

4 Basic properties of a conditional probability

Theorem 2 *Let $p(a|x)$ be a version of conditional probability, then there almost everywhere holds*

- *i) $p(0|x) = 0$, $p(1|x) = 1$,*
- *ii) $0 \leq p(a|x) \leq 1$, $a \in D$*
- *iii) if $a_n \nearrow a \Longrightarrow p(a_n|x) \to p(a|x)$.*

Proof. For every $B \in J$,

$$\int_B p(0|x) d\lambda_F = m(0 * x(B)) = m(0) = 0.$$

Now let $p(1|x) = 1$, then

$$\int_B p(1|x) d\lambda_F = m(1 * x(B)) = m(x(B)) = \int_R 1 d\lambda_F = 1.$$

We prove the second property

$$0 = m(0) = m(a * x(\emptyset)) \leq m(a * x(B)) = \int_B p(a|x) d\lambda_F \leq \int_R p(a|x) d\lambda_F = m(1) = 1.$$

Consider the sets A_0, A_1, where

$$A_0 = \{t \in R; p(a|x) < 0\},$$

$$A_1 = \{t \in R; p(a|x) > 1\},$$

then measure λ_F of the sets A_0, A_1 is equal 0. For example, let $\lambda_F(A_0) > 0$, then

$$\int_{A_0} p(a|x)d\lambda_F < \int_{A_1} 0 d\lambda_F = 0,$$

what is contradiction. And finally, third property, let $a_n \nearrow a$ then

$$m(a * x(B)) \nearrow m(a * x(B)).$$

Therefore

$$\int_B lim_{n\to\infty} p(a_n|x)d\lambda_F = lim_{n\to\infty} p(a_n|x)d\lambda_F = lim_{n\to\infty} m(a_n * x(B)) = m(a * x(B)) = \int_B p(a|x)d\lambda_F.$$

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