ON INTUITIONISTIC FIZZY NEGATIONS AND INTUITIONISTIC FIZZY MODAL OPERATORS

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Abstract

Some relations between intuitionistic fuzzy negations and intuitionistic fuzzy modal operations (from standard type) are studied.

1   On some previous results

The concept of the Intuitionistic Fuzzy Set (IFS, see [1]) was introduced in 1983 as an extension of Zadeh's fuzzy set. All operations, defined over fuzzy sets were transformed for the IFS case. One of them - operartion "negation" now there is 24 different forms (see [3]. In [1] the relations between the "classical" negation and the two standard modal operators "necessity" and "possibility" are given. Here, we shall study the same relations with the rest negations, defined over IFSs.

In some definitions we shall use functions sg and $\overline{sg}$:

$$ \mathrm{sg}(x) = \begin{cases} 1 & \text{if } x > 0 \\ 0 & \text{if } x \le 0 \end{cases}, $$

$$ \overline{sg}(x) = \begin{cases} 0 & \text{if } x > 0 \\ 1 & \text{if } x \le 0 \end{cases} $$

For two IFSs A and B the following relations are valid:

$$ A \subset B \text{ iff } (\forall x \in E)(\mu_A(x) \le \mu_B(x)\nu_A(x) \ge \nu_B(x)), $$

$$ A \supset B \text{ iff } B \subset A, $$

$$ A = B \text{ iff } (\forall x \in E)(\mu_A(x) = \mu_B(x) \& \nu_A(x) = \nu_B(x)). $$

Let A be a fixed IFS. In [1] definitions of standard modal operators are given:

$$ \Box A = \{\langle x, \mu_A(x), 1 - \mu_A(x)\rangle | x \in E\}, $$

$$ \Diamond A = \{\langle x, 1 - \nu_A(x), \nu_A(x)\rangle | x \in E\}. $$

2 Main results

In [3, 4, 5, 6] the following 24 different negations are described.

$$\neg_1 A = \{\langle b, a \rangle | x \in E\},$$

$$\neg_2 A = \{\langle \overline{sg}(a), \mathrm{sg}(a) \rangle | x \in E\},$$

$$\neg_3 A = \{\langle b, a.b + a^2 \rangle | x \in E\},$$

$$\neg_4 A = \{\langle b, 1 - b \rangle | x \in E\},$$

$$\neg_5 A = \{\langle \overline{sg}(1 - b), \mathrm{sg}(1 - b) \rangle | x \in E\},$$

$$\neg_6 A = \{\langle \overline{sg}(1 - b), \mathrm{sg}(a) \rangle | x \in E\},$$

$$\neg_7 A = \{\langle \overline{sg}(1 - b), a \rangle | x \in E\},$$

$$\neg_8 A = \{\langle 1 - a, a \rangle | x \in E\},$$

$$\neg_9 A = \{\langle \overline{sg}(a), a \rangle | x \in E\},$$

$$\neg_{10} A = \{\langle \overline{sg}(1 - b), 1 - b \rangle | x \in E\},$$

$$\neg_{11} A = \{\langle \mathrm{sg}(b), \overline{sg}(b) \rangle | x \in E\},$$

$$\neg_{12} A = \{\langle b(a + b), a(a + b^2) \rangle | x \in E\},$$

$$\neg_{13} A = \{\langle \mathrm{sg}(1 - b), \overline{sg}(1 - a) \rangle | x \in E\},$$

$$\neg_{14} A = \{\langle \mathrm{sg}(b), \overline{sg}(1 - a) \rangle | x \in E\},$$

$$\neg_{15} A = \{\langle \overline{sg}(1 - b), \overline{sg}(1 - a) \rangle | x \in E\},$$

$$\neg_{16} A = \{\langle \overline{sg}(a), \overline{sg}(1 - a) \rangle | x \in E\},$$

$$\neg_{17} A = \{\langle \overline{sg}(1 - b), \overline{sg}(b) \rangle | x \in E\},$$

$$\neg_{18} A = \{\langle x, \nu_A(x).\mathrm{sg}(\mu_A(x)), \mu_A(x).\mathrm{sg}(\nu_A(x)) \rangle | x \in E\},$$

$$\neg_{19} A = \{\langle x, \nu_A(x).\mathrm{sg}(\mu_A(x)), 0 \rangle | x \in E\},$$

$$\neg_{20} A = \{\langle x, \nu_A(x), 0 \rangle | x \in E\},$$

$$\neg_{21} A = \{\langle x, \nu_A(x), \mu_A(x).\nu_A(x) + \mu_A(x)^n \rangle | x \in E\},$$

where real number $n \in [2, \infty)$,

$$\neg_{22} A = \{\langle x, \nu_A(x), \mu_A(x).\nu_A(x) + \overline{sg}(1 - \mu_A(x)) \rangle | x \in E\},$$

$$\neg^{\varepsilon} A = \{\langle x, \min(1, \nu_A(x) + \varepsilon), \max(0, \mu_A(x) - \varepsilon) \rangle | x \in E\},$$

where $\varepsilon \in [0, 1]$,

$$\neg^{\varepsilon, \eta} A = \{\langle x, \min(1, \nu_A(x) + \varepsilon), \max(0, \mu_A(x) - \eta) \rangle | x \in E\},$$

where $0 \leq \varepsilon \leq \eta \leq 1$.

Now, following and extending the idea from [8] we shall prove the following
Theorem 1: For every IFS A the following properties are valid:
(1) $\neg_1 \square A = \Diamond \neg_1 A,$
(2) $\neg_1 \Diamond A = \square \neg_1 A,$
(3) $\neg_2 \square A = \square \neg_2 A,$
(4) $\neg_2 \Diamond A \subset \Diamond \neg_2 A,$
(5) $\neg_3 \square A \supset \square \neg_3 A,$

(6) $\neg_3 \Diamond A \subset \Diamond \neg_3 A$,

(7) $\neg_4 \Box A \supset \Box \neg_4 A$,

(8) $\neg_4 \Diamond A = \Diamond \neg_4 A$,

(9) $\neg_5 \Box A \supset \Box \neg_5 A$,

(10) $\neg_5 \Diamond A = \Diamond \neg_5 A$,

(11) $\neg_6 \Box A \supset \Box \neg_6 A$,

(12) $\neg_6 \Diamond A = \Diamond \neg_6 A$,

(13) $\neg_7 \Box A \supset \Box \neg_7 A$,

(14) $\neg_7 \Diamond A \subset \Diamond \neg_7 A$,

(15) $\neg_8 \Box A = \Box \neg_8 A$,

(16) $\neg_8 \Diamond A \subset \Diamond \neg_8 A$,

(17) $\neg_9 \Box A \supset \Box \neg_9 A$,

(18) $\neg_9 \Diamond A \subset \Diamond \neg_9 A$,

(19) $\neg_{10} \Box A \supset \Box \neg_{10} A$,

(20) $\neg_{11} \Diamond A = \Diamond \neg_{11} A$,

(21) $\neg_{13} \Box A = \Box \neg_{12} A$,

(22) $\neg_{15} \Box A \supset \Box \neg_{15} A$,

(23) $\neg_{15} \Diamond A \subset \Diamond \neg_{15} A$,

(24) $\neg_{16} \Box A \supset \Box \neg_{16} A$,

(25) $\neg_{17} \Box A \supset \Box \neg_{17} A$,

(26) $\neg_{17} \Diamond A \subset \Diamond \neg_{17} A$,

(27) $\neg_{18} \Box A \supset \Box \neg_{18} A$,

(28) $\neg_{18} \Diamond A \subset \Diamond \neg_{18} A$,

(29) $\neg_{19} \Box A \supset \Box \neg_{19} A$,

(30) $\neg_{19} \Diamond A \subset \Diamond \neg_{19} A$,

(31) $\neg_{20} \Box A \supset \Box \neg_{20} A$,

(32) $\neg_{20} \Diamond A \subset \Diamond \neg_{20} A$,

(33) $\neg_{21} \Box A \supset \Box \neg_{21} A$,

(34) $\neg_{21} \Diamond A \subset \Diamond \neg_{21} A$,

(35) $\neg_{22} \Box A \supset \Box \neg_{22} A$,

(36) $\neg_{22} \Diamond A \subset \Diamond \neg_{22} A$,

(37) $\neg^\varepsilon \Box A \supset \Box \neg^\varepsilon A$,

(38) $\neg^\varepsilon \Diamond A \subset \Diamond \neg^\varepsilon A$,

(39) $\neg^{\varepsilon,\eta} \Box A \supset \Box \neg^{\varepsilon,\eta} A$,

(40) $\neg^{\varepsilon,\eta} \Diamond A \subset \Diamond \neg^{\varepsilon,\eta} A$.

Proof: Let us prove, for example (40). The rest assertions can be proved analogously. Let $0 \le \varepsilon \le \eta \le 1$ for some ε and η. Then

$$\neg^{\varepsilon,\eta} \Diamond A = \neg^{\varepsilon,\eta} \{\langle x, 1 - \nu_A(x), \nu_A(x)\rangle | x \in E\}$$

$$= \{\langle x, \min(1, \nu_A(x) + \varepsilon), \max(0, 1 - \nu_A(x) - \eta)\rangle | x \in E\}.$$

$$\Diamond \neg^{\varepsilon,\eta} A = \Diamond \{\langle x, \min(1, \nu_A(x) + \varepsilon), \max(0, \mu_A(x) - \eta)\rangle | x \in E\}$$

$$= \{\langle x, 1 - \max(0, \mu_A(x) - \eta), \max(0, \mu_A(x) - \eta)\rangle | x \in E\}.$$

Let

$$X \equiv 1 - \max(0, \mu_A(x) - \eta) - \min(1, \nu_A(x) + \varepsilon).$$

If $\nu_A(x) + \varepsilon \ge 1$, then

$$\mu_A(x) - \eta \le 1 - \nu_A(x) - \eta \le \varepsilon - \eta \le 0$$

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and $X = 1 - 1 - 0 = 0$. If $\nu_A(x) + \varepsilon \leq 1$, then there are two subcases. If $\mu_A(x) - \eta \leq 0$, then

$$X = 1 - (\nu_A(x) + \varepsilon) - 0 \geq 0$$

and if $\mu_A(x) - \eta \geq 0$, then

$$X = 1 - (\nu_A(x) + \varepsilon) - \mu_A(x) + \eta = 1 - \mu_A(x) - \mu_A(x) + \eta - \varepsilon \geq 0.$$

Therefore, the first component of the second trm is higher than the first component of the first term, while the inequality

$$\max(0, 1 - \nu_A(x) - \eta) - \max(0, \mu_A(x) - \eta) \geq 0$$

is obvious. Therefore inclusion (40) is valid.

In [2] there were shown cases in which some intuitionistic fuzzy (non-classical) negations do not satisfy De Morgan's laws. Now, by analoy with this result, we shall study the De Morgans' form of modal logic operators (see, e.g. [7]):

$$\square A = \neg \Diamond \neg A, \quad \Diamond A = \neg \square \neg A$$

and will formulate the following assertion that is proved as above one.

Theorem 2: For every IFS A the following properties are valid:

(1) $\neg_1 \square \neg_1 A = \Diamond A$,

(2) $\neg_1 \Diamond \neg_1 A = \square A$,

(3) $\neg_3 \square \neg_3 A = \Diamond A$,

(4) $\neg_4 \square \neg_4 A = \Diamond A$,

(5) $\neg_4 \Diamond \neg_4 A \supset \square A$,

(6) $\neg_7 \Diamond \neg_7 A \subset \square A$,

(7) $\neg_8 \Diamond \neg_8 A = \square A$,

(8) $\neg_9 \Diamond \neg_9 A \subset \square A$.

3 Conclusion

In a next research authors will study the above properties for the case of extended intuitionistic fuzzy modal and topological operators.

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