A method for constructing non-t-representable intuitionistic fuzzy t-norms satisfying the residuation principle

Glad Deschrijver and Etienne E. Kerre
Department of Mathematics and Computer Science, Ghent University
Fuzziness and Uncertainty Modelling Research Unit
Krijgslaan 281 (S9), B-9000 Gent, Belgium
Email: {glad.deschrijver|etienne.kerre}@rug.ac.be
Homepage: http://fuzzy.rug.ac.be

Abstract

Intuitionistic fuzzy sets constitute an extension of fuzzy sets giving both a membership degree and a non-membership degree, whose sum must be smaller than or equal to 1. In fuzzy set theory, an important class of triangular norms is the class of those that satisfy the residuation principle. In this paper we give a construction for intuitionistic fuzzy t-norms satisfying the residuation principle which are not t-representable.

Keywords: intuitionistic fuzzy set, intuitionistic fuzzy t-norm, residuation principle

1 Introduction

An important notion in fuzzy set theory is that of triangular norms and conorms: t-norms and t-conorms are used to define a generalized intersection and union of fuzzy sets, to extend the composition of fuzzy relations e.g. for use in approximate reasoning, and for many other purposes. An important class of t-norms is the class of t-norms that satisfy the residuation principle (see e.g. [9, 12]). In non-classical logics they make it possible to define the implication as the residuum of the conjunction, the latter being modelled by a t-norm satisfying the residuation principle [11, 12, 13]. If the induced negator is involutive, then the disjunction can be defined as the dual of the conjunction (by the de Morgan law). Triangular norms satisfying the residuation principle are also useful in fuzzy preference modelling [14] (where the preferences of individuals between two alternatives are described by a real number in $[0, 1]$). In this paper we give a construction method, using left-continuous t-norms, for intuitionistic fuzzy t-norms satisfying the residuation principle but which are not t-representable.

2 Preliminary definitions

Intuitionistic fuzzy sets were introduced by Atanassov in 1983 and are defined as follows.

Definition 2.1 [1, 2, 3] *An intuitionistic fuzzy set A in a universe U is an object*

$$A = \{(u, \mu_A(u), \nu_A(u)) \mid u \in U\},$$

where, for all $u \in U$, $\mu_A(u) \in [0, 1]$ and $\nu_A(u) \in [0, 1]$ are called the membership degree and the non-membership degree, respectively, of u in A, and furthermore satisfy $\mu_A(u) + \nu_A(u) \leq 1$.

Deschrijver and Kerre [8] have shown that intuitionistic fuzzy sets can also be seen as L-fuzzy sets in the sense of Goguen [10], w.r.t. the complete lattice $(L^*, \leq_{L^*})$ defined by [8]:

$$L^* = \{(x_1, x_2) \mid (x_1, x_2) \in [0, 1]^2 \text{ and } x_1 + x_2 \leq 1\},$$

$$(x_1, x_2) \leq_{L^*} (y_1, y_2) \Leftrightarrow x_1 \leq y_1 \text{ and } x_2 \geq y_2,$$
$$\forall (x_1, x_2), (y_1, y_2) \in L^*.$$

We denote its units by $0_{L^*} = (0, 1)$ and $1_{L^*} = (1, 0)$.

Note that if, for $x = (x_1, x_2), y = (y_1, y_2) \in L^*$, $x_1 < y_1$ and $x_2 < y_2$, or $x_1 > y_1$ and $x_2 > y_2$, then x and y are incomparable w.r.t $\leq_{L^*}$, denoted as $x \|_{L^*} y$.
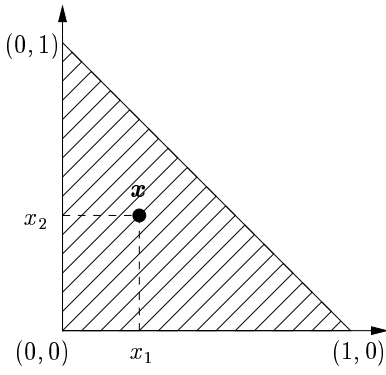
Figure 1: Graphical representation of the set L^*

The shaded area in Figure 1 is the set of elements $x = (x_1, x_2)$ belonging to L^*. From now on, we will assume that if $x \in L^*$, then x_1 and x_2 denote respectively the first and the second component of x, i.e. $x = (x_1, x_2)$. We also define the following set for further usage: $D = \{x \mid x \in L^*$ and $x_1 + x_2 = 1\}$.

Using this lattice, we easily see that with every intuitionistic fuzzy set $A = \{(u, \mu_A(u), \nu_A(u)) \mid u \in U\}$ corresponds an L^*-fuzzy set, i.e. a mapping $A : U \to L^* : u \mapsto (\mu_A(u), \nu_A(u))$.

Definition 2.2 *[5, 6] An intuitionistic fuzzy t-norm is a commutative, associative, increasing $(L^*)^2 - L^*$ mapping $\mathcal{T}$ satisfying $\mathcal{T}(1_{L^*}, x) = x$, for all $x \in L^*$.*

Let T be a t-norm and S a t-conorm, and assume that $T(a, b) \leq 1 - S(1 - a, 1 - b)$, for all $a, b \in [0, 1]$. Then the mapping $\mathcal{T}$ defined by $\mathcal{T}(x, y) = (T(x_1, y_1), S(x_2, y_2))$, for all $x, y \in L^*$, is an intuitionistic fuzzy t-norm.

Definition 2.3 *[5, 6] An intuitionistic fuzzy t-norm $\mathcal{T}$ is called t-representable iff there exist a t-norm T and a t-conorm S on $[0, 1]$ such that, for all $x, y \in L^*$,*

$$\mathcal{T}(x, y) = (T(x_1, y_1), S(x_2, y_2)).$$

Not all intuitionistic fuzzy t-norms are t-representable. For instance the intuitionistic fuzzy Lukasiewicz t-norm $\mathcal{T}_W$ defined by [5, 6]

$$\mathcal{T}_W(x, y) = (\max(0, x_1 + y_1 - 1),$$
$$\min(1, x_2 + 1 - y_1, y_2 + 1 - x_1))$$

is an intuitionistic fuzzy t-norm which is not t-representable.

Definition 2.4 *An intuitionistic fuzzy negator is a decreasing $L^* - L^*$ mapping $\mathcal{N}$ satisfying $\mathcal{N}(0_{L^*}) = 1_{L^*}$ and $\mathcal{N}(1_{L^*}) = 0_{L^*}$. If $\mathcal{N}(\mathcal{N}(x)) = x$, for all $x \in L^*$, then $\mathcal{N}$ is called an involutive intuitionistic fuzzy negator.*

3 The residuation principle

The notion of residuation principle for intuitionistic fuzzy t-norms is an extension of the same notion for t-norms on $[0, 1]$.

Definition 3.1 *[6] An intuitionistic fuzzy t-norm $\mathcal{T}$ satisfies the residuation principle if and only if, for all $x, y, z \in L^*$,*

$$\mathcal{T}(x, y) \leq_{L^*} z \Leftrightarrow y \leq_{L^*} \mathcal{I}_{\mathcal{T}}(x, z),$$

where $\mathcal{I}_{\mathcal{T}}$ denotes the residual implicator generated by $\mathcal{T}$, defined as, for $x, y \in L^*$,

$$\mathcal{I}_{\mathcal{T}}(x, y) = \sup\{\gamma \mid \gamma \in L^* \text{ and } \mathcal{T}(x, \gamma) \leq_{L^*} y\}.$$

For further usage, we define the intuitionistic fuzzy negator $\mathcal{N}_{\mathcal{T}}$ induced by $\mathcal{I}_{\mathcal{T}}$ as $\mathcal{N}_{\mathcal{T}}(x) = \mathcal{I}_{\mathcal{T}}(x, 0_{L^*})$, for all $x \in L^*$.

4 Construction of non-t-representable intuitionistic fuzzy t-norms satisfying the residuation principle

In [7] we have proven the following.

Theorem 4.1 *Let T be a left-continuous t-norm and $t \in [0, 1]$. Define an $(L^*)^2 \to L^*$ mapping $\mathcal{T}_{T,t}$ by, for all $x, y \in L^*$,*

$$\mathcal{T}_{T,t}(x, y) = (T(x_1, y_1),$$
$$\min\{1 - T(1 - t, T(1 - x_2, 1 - y_2)),$$
$$1 - T(1 - y_2, x_1), 1 - T(1 - x_2, y_1)\}).$$

Then $\mathcal{T}_{T,t}$ is an intuitionistic fuzzy t-norm satisfying the residuation principle.

Note that $\mathcal{T}_{T,t}$ is t-representable if and only if $t = 0$. If $t = 0$, then clearly $\mathcal{T}_{T,0}(x, y) = (T(x_1, y_1), 1 - T(1 - x_2, 1 - y_2))$. If $t \neq 0$, then $\mathcal{T}_{T,t}((0, 0), (0, 0)) = (0, t)$, so $\mathcal{T}_{T,t}$ is

not t-representable, since for any t-representable intuitionistic fuzzy t-norm $\mathcal{T}$ it holds that $\mathcal{T}((0,0),(0,0)) = (0,0)$.

Using Theorem 4.1 we can construct intuitionistic fuzzy t-norms satisfying the residuation principle, starting from left-continuous t-norms on $[0,1]$. For example, if T is the Lukasiewicz t-norm T_W defined by $T_W(a,b) = \max(0, a+b-1)$, for all $a,b \in [0,1]$, and $t = 1$, then we obtain $\mathcal{T}_{T_W,1} = \mathcal{T}_W$.

In [7] it is shown that the residual implicator of $\mathcal{T}_{T,t}$ is given by

$$\mathcal{I}_{\mathcal{T}_{T,t}}(x,y) = (\min\{I_T(x_1,y_1), I_T(1-x_2, 1-y_2)\},$$
$$\max\{1 - I_T(T(1-x_2, 1-t),$$
$$1-y_2), 1 - I_T(x_1, 1-y_2)\}), \qquad (1)$$

where I_T is the residual implicator of T given by $I_T(a,b) = \sup\{c \mid c \in [0,1] \text{ and } T(a,c) \le b\}$. The intuitionistic fuzzy negator $\mathcal{N}_{\mathcal{T}_{T,t}}$ induced by $\mathcal{I}_{\mathcal{T}_{T,t}}$ is given by [7]

$$\mathcal{N}_{\mathcal{T}_{T,t}}(x) = \mathcal{I}_{\mathcal{T}_{T,t}}(x, 0_{L^*})$$
$$= (N_T(1-x_2), \max\{1 - N_T(T(1-x_2,$$
$$1-t)), 1 - N_T(x_1)\}),$$

where N_T is the fuzzy negator induced by I_T, i.e. $N_T(a) = I_T(a,0)$, for all $a \in [0,1]$. In [7] it is proven that $\mathcal{N}_{\mathcal{T}_{T,t}}$ is involutive if and only if $t = 1$ and N_T is involutive.

In [13], Smets and Magrez outlined an axiom scheme for implicators on $[0,1]$. We introduced similar axioms for intuitionistic fuzzy implications in [4].

Definition 4.1 (Axioms of Smets and Magrez for an intuitionistic fuzzy implicator $\mathcal{I}$) *For all $x,y,z \in L^*$:*

(A.1) $\mathcal{I}(.,y)$ *is decreasing in* L^*,
  $\mathcal{I}(x,.)$ *is increasing in* L^*;

(A.2) $\mathcal{I}(1_{L^*}, x) = x$;

(A.3) $\mathcal{I}(x,y) = \mathcal{I}(\mathcal{N}_{\mathcal{I}}(y), \mathcal{N}_{\mathcal{I}}(x))$;

(A.4) $\mathcal{I}(x, \mathcal{I}(y,z)) = \mathcal{I}(y, \mathcal{I}(x,z))$;

(A.5) $x \le_{L^*} y \Leftrightarrow \mathcal{I}(x,y) = 1_{L^*}$;

(A.6) $\mathcal{I}$ *is a continuous* $(L^*)^2 \to L^*$ *mapping*.

We now check under which conditions $\mathcal{I}_{\mathcal{T}_{T,t}}$ satisfies the Smets–Magrez axioms. In [4] it is shown that the residual implicator $\mathcal{I}_{\mathcal{T}}$ of any intuitionistic fuzzy t-norm $\mathcal{T}$ satisfies (A.2).

Theorem 4.2 $\mathcal{I}_{\mathcal{T}_{T,t}}$ *satisfies (A.3) if and only if $t = 1$ and I_T satisfies (A.3).*

Theorem 4.3 $\mathcal{I}_{\mathcal{T}_{T,t}}$ *satisfies (A.4) if and only if I_T satisfies (A.4) and*

$$I_T(1-x_2, I_T(1-t, 1-z_2))$$
$$= I_T(T(1-x_2, 1-t), 1-z_2). \qquad (2)$$

Theorem 4.4 $\mathcal{I}_{\mathcal{T}_{T,t}}$ *satisfies (A.5) if and only if I_T satisfies (A.5).*

Clearly, if T is continuous and I_T satisfies (A.6), then $\mathcal{I}_{\mathcal{T}_{T,t}}$ satisfies (A.6).

Let us consider for example the nilpotent minimum T_{M_0} defined in [9] by, for $a,b \in [0,1]$:

$$T_{M_0}(a,b) = \begin{cases} \min(a,b), & \text{if } b > N_0(a), \\ 0, & \text{otherwise,} \end{cases}$$

for any involutive negator N_0. The corresponding residual implicator and S-implicator coincide and are equal to [9]:

$$I_{T_{M_0}}(a,b) = \begin{cases} 1, & \text{if } a \le b, \\ \max(N_0(a), b), & \text{otherwise.} \end{cases}$$

Moreover $I_{T_{M_0}}(a,0) = N_0(a)$, for all $a \in [0,1]$, and $I_{T_{M_0}}$ satisfies (A.2), (A.3), (A.4) and (A.5).

We obtain (denote $\mathcal{T}_{T_{M_0},1}$ by $\mathcal{T}_{M_0}$)

$$\mathcal{T}_{M_0}(x,y) = \mathcal{T}_{T_{M_0},1}(x,y)$$
$$= \begin{cases} (\min(x_1,y_1), \min\{1 - \min(1-y_2, x_1), \\ \quad 1 - \min(1-x_2, y_1)\}), \text{ if } y_1 > N_0(x_1); \\ (0, \min\{1 - \min(1-y_2, x_1), \\ \quad 1 - \min(1-x_2, y_1)\}), \text{ if } y_1 \le N_0(x_1) < \\ \quad 1 - y_2 \text{ and } N_0(y_1) < 1 - x_2; \\ (0, 1 - \min(1-x_2, y_1)), \text{ if } 1 - y_2 \le N_0(x_1) \\ \quad \text{and } N_0(y_1) < 1 - x_2; \\ (0, 1 - \min(1-y_2, x_1)), \text{ if } 1 - x_2 \le N_0(y_1) \\ \quad \text{and } N_0(x_1) < 1 - y_2; \\ 0_{L^*}, \quad \text{otherwise.} \end{cases}$$

Let $\mathcal{T}_{M_0}^*$ be the dual intuitionistic fuzzy t-conorm of $\mathcal{T}_{M_0}$ w.r.t. $\mathcal{N}_0 = \mathcal{N}_{\mathcal{T}_{M_0}}$, i.e. $\mathcal{T}_{M_0}^*(x,y) =$

$\mathcal{N}_0(\mathcal{T}_{M_0}(\mathcal{N}_0(x), \mathcal{N}_0(y)))$. Note that from the above it follows that $\mathcal{N}_0(x) = (N_0(1 - x_2), 1 - N_0(x_1))$. Then the S-implicator $\mathcal{I}_{\mathcal{T}_{M_0}^*, \mathcal{N}_0}$ induced by $\mathcal{T}_{M_0}^*$ and $\mathcal{N}_0$ is given by

$$
\begin{aligned}
&\mathcal{I}_{\mathcal{T}_{M_0}^*, \mathcal{N}_0}(x, y) = \mathcal{T}_{M_0}^*(\mathcal{N}_0(x), y) \\
&= \mathcal{N}_0(\mathcal{T}_{M_0}(x, \mathcal{N}_0(y))) \\
&= \begin{cases}
(N_0(\max\{\min(N_0(y_1), x_1), \min(1 - x_2, \\
\quad N_0(1 - y_2))\}), 1 - N_0(\min(x_1, \\
\quad N_0(1 - y_2)))), \text{ if } 1 - y_2 < x_1; \\
(N_0(\max\{\min(N_0(y_1), x_1), \min(1 - x_2, \\
\quad N_0(1 - y_2))\}), 0), \text{ if } 1 - y_2 \geq x_1 > y_1 \\
\quad \text{and } y_2 > x_2; \\
(N_0(\min(1 - x_2, N_0(1 - y_2))\}), 0), \\
\quad \text{if } x_1 \leq y_1 \text{ and } y_2 > x_2; \\
(N_0(\min(N_0(y_1), x_1)\}), 0), \\
\quad \text{if } x_2 \geq y_2 \text{ and } x_1 > y_1; \\
1_{L^*}, \text{ otherwise.}
\end{cases}
\end{aligned}
$$

On the other hand, using (1) we obtain that $\mathcal{I}_{T_{M_0}}(x, y) = \mathcal{I}_{\mathcal{T}_{M_0}^*, \mathcal{N}_0}(x, y)$, for all $x, y \in L^*$.

From the above and from Theorem 4.2, Theorem 4.3 and Theorem 4.4 it follows that $\mathcal{I}_{\mathcal{T}_{M_0}}$ satisfies (A.2), (A.3), (A.4) and (A.5).

References

[1] K. T. Atanassov, Intuitionistic fuzzy sets, 1983, VII ITKR's Session, Sofia (deposed in Central Sci.-Technical Library of Bulg. Acad. of Sci., 1697/84) (in Bulgarian).

[2] K. T. Atanassov, Intuitionistic fuzzy sets, *Fuzzy Sets and Systems* **20** (1986), 87–96.

[3] K. T. Atanassov, *Intuitionistic fuzzy sets*, Physica-Verlag, Heidelberg/New York, 1999.

[4] C. Cornelis, G. Deschrijver and E. E. Kerre, Implication in intuitionistic and interval-valued fuzzy set theory: construction, classification, application, 2002, Submitted to the Journal of Approximate Reasoning.

[5] G. Deschrijver, C. Cornelis and E. E. Kerre, Intuitionistic fuzzy connectives revisited, In *Proceedings of the 9th International Conference on Information Processing and Management of Uncertainty in Knowledge-Based Systems*, 2002, pp. 1839–1844.

[6] G. Deschrijver, C. Cornelis and E. E. Kerre, On the representation of intuitionistic fuzzy t-norms and t-conorms, 2002, Accepted to IEEE Transactions on Fuzzy Systems.

[7] G. Deschrijver and E. E. Kerre, Classes of intuitionistic fuzzy t-norms satisfying the residuation principle, 2003, Submitted to the International Journal of Uncertainty, Fuzziness and Knowledge-Based Systems.

[8] G. Deschrijver and E. E. Kerre, On the relationship between some extensions of fuzzy set theory, *Fuzzy Sets and Systems* **133**, 2 (2003), 227–235.

[9] J. C. Fodor, Contrapositive symmetry of fuzzy implications, *Fuzzy Sets and Systems* **69** (1995), 141–156.

[10] J. Goguen, *L*-fuzzy sets, *J. Math. Anal. Appl.* **18** (1967), 145–174.

[11] P. Hájek, *Metamathematics of fuzzy logic*, Kluwer Academic Publishers, Dordrecht, 1998.

[12] E. P. Klement and M. Navara, A survey on different triangular norm-based fuzzy logics, *Fuzzy Sets and Systems* **101** (1999), 241–251.

[13] P. Smets and P. Magrez, Implication in fuzzy logic, *International Journal of Approximate Reasoning* **1** (1987), 327–347.

[14] B. Van de Walle, B. De Baets and E. E. Kerre, A plea for the use of Lukasiewicz triplets in the definition of fuzzy preference structures (I). General argumentation, *Fuzzy Sets and Systems* **97** (1998), 349–359.