INTUITIONISTIC FUZZY IMPLICATION $\rightarrow^{\varepsilon,\eta}$ AND INTUITIONISTIC FUZZY NEGATION $\neg^{\varepsilon,\eta}$

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Abstract

Up to now a series of intuitionistic fuzzy implications and negations were constructed. In the present paper a new implication and negation are described and some of their specific properties are discussed. The relations between it and the other negations are studied. The standard and modified Laws for Excluded Middle, the standard and modified De Morgan's Laws, and the G. Klir and Bo Yuan's axioms are checked for the new implication and negation.

1 Introduction: On some previous results

Variants of intuitionistic fuzzy implications are discussed in [3, 4, 5, 11, 12]. In [6] they are the basis for obtaining intuitionistic fuzzy negations. There, some properties of these implications and negations are studied.

Let x be a variable. Then its intuitionistic fuzzy truth-value is represented by the ordered couple

$$V(x) = \langle a, b\rangle,$$

so that $a, b, a + b \in [0, 1]$, where a and b are degrees of validity and of non-validity of x. Any other formula is estimated by analogy.

Obviously, when V is ordinary fuzzy truth-value estimation, we have

$$b = 1 - a.$$

Everywhere below we shall assume that for the three variables x, y and z equalities: $V(x) = \langle a, b\rangle, V(y) = \langle c, d\rangle, V(z) = \langle e, f\rangle$ $(a, b, c, d, e, f, a + b, c + d, e + f \in [0, 1])$ hold.

For the needs of the discussion below we shall define the notion of Intuitionistic Fuzzy Tautology (IFT, see, [1, 2]) by:

$$x \text{ is an IFT if and only if } a \geq b,$$

while x will be a tautology iff $a = 1$ and $b = 0$.

For two variables x and y operations "conjunction" (&) and "disjunction" ($\vee$) are defined by:

$$V(x \& y) = \langle \min(a, c), \max(b, d)\rangle,$$

$$V(x \vee y) = \langle \max(a, c), \min(b, d)\rangle.$$

In [5] (that is based on [14]), the first 15 different intutionistic fuzzy implications are introduced. For each one of these implications in [6] we constructed the respective negations, using as a basis equality

$$\neg x = x \rightarrow 0$$

or

$$\neg\langle a,b\rangle = \langle a,b\rangle \to \langle 0,1\rangle.$$

Part of the new negations coincide, but five of them are different. The first one is $\neg_1\langle a,b\rangle = \langle b,a\rangle$. For these negations and for their corresponding implications the following three properties are checked in [4]:

Property P1: $A \to \neg\neg A$,

Property P2: $\neg\neg A \to A$,

Property P3: $\neg\neg\neg A = \neg A$.

Obviously, negation $\neg_1$ is a classical negation (it satisfy simultaneously properties P1 and P2), while for the four other ones is shown that they have intuitionistic (in the sense of Brouwer and Heyting (see, e.g. [13, 15]) behaviour (they satisfy property P1 and do not satisfy property P2). All negations satisfy property P3.

In [7] the validity of the Law for Excluded Middle (LEM) in the following forms is studied:

$$\langle a,b\rangle \vee \neg\langle a,b\rangle = \langle 1,0\rangle$$

(tautology-form) and

$$\langle a,b\rangle \vee \neg\langle a,b\rangle = \langle p,q\rangle,$$

where $p,q \in [0,1], p+q \geq 1$ and (the specific condition) $p \geq q$ (IFT-form) and a Modified LEM in the forms:

$$\neg\neg\langle a,b\rangle \vee \neg\langle a,b\rangle = \langle 1,0\rangle$$

(tautology-form) and

$$\neg\neg\langle a,b\rangle \vee \neg\langle a,b\rangle = \langle p,q\rangle,$$

(IFT-form).

Usually, De Morgan's Laws have the forms:

$$\neg x \wedge \neg y = \neg(x \vee y), \tag{1}$$

$$\neg x \vee \neg y = \neg(x \wedge y). \tag{2}$$

The above mentioned change of the Law for Excluded Middle inspired the idea from [7] to study the validity of De Morgan's Laws that the classical negation $\neg_1$ satisfies. Really, it can be easily proved that the expressions

$$\neg_1(\neg_1 x \vee \neg_1 y) = x \wedge y \tag{3}$$

and

$$\neg_1(\neg_1 x \wedge \neg_1 y) = x \vee y \tag{4}$$

are IFTs, but the other negations do not satisfy these equalities. For them the following assertion is valid for every two propositional forms x and y:

$$\neg_i(\neg_i x \vee \neg_i y) = \neg_i\neg_i x \wedge \neg_i\neg_i y \tag{5}$$

$$\neg_i(\neg_i x \wedge \neg_i y) = \neg_i\neg_i x \vee \neg_i\neg_i y \tag{6}$$

for $i = 2,4,5$, while negation $\neg_3$ does not satisfies these equalities.

The idea for the modifications of Law for Excluded Middle and of De Morgan's Laws give us possibility to introduce new implications (see, e.g. [8]) with the forms:

$$x \to y = \neg x \vee y \tag{7}$$

and

$$x \to y = \neg x \vee \neg\neg y. \tag{8}$$

2 Main results

Here we shall introduce a set of new negations. They are related to the previous ones. They generalize the classical negation, but on the other hand, they have some non-classical properties. The set has the form

$$\mathcal{N} = \{\neg^{\varepsilon,\eta} \mid 0 \leq \varepsilon < 1 \ \&\ 0 \leq \eta < 1\},$$

where for $a, b \in [0,1]$ and $a + b \leq 1$:

$$\neg^{\varepsilon,\eta}\langle a, b\rangle = \langle \min(1, b+\varepsilon), \max(0, a-\eta)\rangle.$$

Below we shall study some basic properties of an arbitrary element of $\mathcal{N}$.

For ε and η there are three cases.

2.1 $\eta < \varepsilon$

This case is impossible, because, e.g.

$$\neg^{0.4,0.2}\langle 0.5, 0.4\rangle = \langle \min(1, 0.8), \max(0, 0.3)\rangle = \langle 0.8, 0.3\rangle$$

that is incorrect from IFS-point of view.

2.2 $\eta = \varepsilon$

It is discussed in [9, 10].

2.3 $\eta > \varepsilon$

Let everywhere below $0 \leq \varepsilon < \eta < 1$ be fixed.

First, we see, that couple $\langle \min(1, b+\varepsilon), \max(0, a-\eta)\rangle$ is intuitionistic fuzzy one. Really, if $a - \eta \leq 0$, then

$$\min(1, b+\varepsilon) + \max(0, a-\eta) = \min(1, b+\varepsilon) \leq 1.$$

If $a - \eta > 0$, i.e., $a > \eta$, then $b + \varepsilon < a + b \leq 1$ and

$$\min(1, b+\varepsilon) + \max(0, a-\varepsilon) \leq b + \varepsilon + a - \varepsilon = a + b \leq 1.$$

Second, let us note that for every $p \in [0,1]$ and $q \geq 0$ the following equation holds:

$$\max(p, \min(1, q)) = \min(1, \max(p, q)) \tag{9}$$

and for every $p \in [0,1]$ and $q \in [-1,1]$ the following equation holds:

$$\min(p, \max(0, q)) = \max(0, \min(p, q)) \qquad (10).$$

By analogy with the above, and using (9) and (10), we can construct two new implications, generated by the new negation. The first of them is based on (7) and has the form:

$$\langle a, b\rangle \rightarrow^{\varepsilon,\eta} \langle c, d\rangle = \langle \max(c, \min(1, b+\varepsilon)), \min(d, \max(0, a-\eta))\rangle$$

$$= \langle \min(1, \max(c, b+\varepsilon)), \max(0, \min(d, a-\eta))\rangle.$$

Now, we see that

$$\neg^{\varepsilon,\eta}\langle a, b\rangle = \langle a, b\rangle \rightarrow^{\varepsilon,\eta} \langle 0, 1\rangle = \langle \min(1, b+\varepsilon), \max(0, a-\eta)\rangle,$$

i.e., the negation generated by implication $\rightarrow^{\varepsilon,\eta}$ coincides with negation $\neg^{\varepsilon,\eta}$.

The second implication that we can construct with negation $\neg^{\varepsilon,\eta}$ is based on (8) and has the form:

$$\langle a, b\rangle \rightarrow^{\varepsilon,\eta} \langle c, d\rangle = \neg^{\varepsilon,\eta}\langle a, b\rangle \vee \neg^{\varepsilon,\eta}\neg^{\varepsilon,\eta}\langle c, d\rangle$$

$$= \langle \min(1, b+\varepsilon), \max(0, a-\eta)\rangle \vee \neg^{\varepsilon,\eta}\langle \min(1, d+\varepsilon), \max(0, c-\eta)\rangle$$

$$= \langle \min(1, b+\varepsilon), \max(0, a-\eta)\rangle \vee \langle \min(1, \max(0, c-\eta)+\varepsilon), \max(0, \min(1, d+\varepsilon)-\eta)\rangle$$

$$= \langle \min(1, b+\varepsilon), \max(0, a-\varepsilon)\rangle \vee \langle \min(1, \max(\varepsilon, c-\eta+\varepsilon)), \max(0, \min(1-\eta, d+\varepsilon-\eta))\rangle$$

$$= \langle \max(\min(1, b+\varepsilon), \min(1, \max(\varepsilon, c-\eta+\varepsilon))), \min(\max(0, a-\varepsilon), \max(0, \min(1-\eta, d+\varepsilon-\eta)))\rangle$$

$$= \langle \min(1, \max(b+\varepsilon, \max(\varepsilon, c-\eta+\varepsilon)), \max(0, \min(a-\varepsilon, 1-\eta, d+\varepsilon-\eta)))\rangle$$

Therefore, the two implications generated by negation $\neg^{\varepsilon,\eta}$ coincide.

Third, we shall formulate similar assertions as in [4], but for the new negation.

Theorem 1: Negation $\neg^{\varepsilon,\eta}$

(a) satisfies Property 1 for its own implication in the case of IFT, but not as a tautology.

(b) satisfies Property 2 for its own implication in the case of IFT, but not as a tautology.

(c) does not satisfy Property 3.

Proof: (a) Let x be a given propositional form.

$$\langle a, b\rangle \rightarrow^{\varepsilon,\eta} \neg^{\varepsilon,\eta}\neg^{\varepsilon,\eta}\langle a, b\rangle$$

$$= \langle a, b\rangle \rightarrow^{\varepsilon,\eta} \neg^{\varepsilon,\eta}\langle \min(1, b+\varepsilon), \max(0, a-\eta)\rangle$$

$$= \langle a, b\rangle \rightarrow^{\varepsilon,\eta} \langle \min(1, \max(\varepsilon, a-\eta+\varepsilon)), \max(0, \min(1-\eta, b+\varepsilon-\eta))\rangle$$

$$= \langle \min(1, \max(\min(1, \max(\varepsilon, a-\eta+\varepsilon)), b+\varepsilon)), \max(0, \min(\max(0, \min(1-\eta, b+\varepsilon-\eta)), a-\eta))\rangle$$

$$= \langle \min(1, \max(\max(\varepsilon, a-\eta+\varepsilon), b+\varepsilon)), \max(0, \min(\max(0, \min(1-\eta, b+\varepsilon-\eta)), a-\eta))\rangle.$$

Obviously, this expression is different than $\langle 1, 0\rangle$, i.e., it is not a tautology.

Let

$$X = \min(1, \max(\max(\varepsilon, a-\eta+\varepsilon), b+\varepsilon)) - \max(0, \min(\max(0, \min(1-\eta, b+\varepsilon-\eta)), a-\eta)).$$

If $a \leq \eta$, then

$$X = \min(1, \max(\varepsilon, b+\varepsilon)) - \max(0, a-\eta)) = \min(1, b+\varepsilon) - 0 \geq 0.$$

If $a > \eta$, then

$$X = \min(1, \max(a-\eta, b+\varepsilon)) - \max(0, \min(\max(0, \min(1-\eta, b+\varepsilon-\eta)), a-\eta)).$$

If $b+\varepsilon < 1$, then, using (10), we obtain

$$X = \min(1, \max(a-\eta, b+\varepsilon)) - \max(0, \min(\max(0, b+\varepsilon-\eta), a-\eta)).$$

$$= \min(1, \max(a-\eta, b+\varepsilon)) - \min(\max(0, b+\varepsilon-\eta), \max(0, a-\eta))$$

$$= \min(1, \max(a-\eta, b+\varepsilon)) - \min(\max(0, b+\varepsilon-\eta), a-\eta)$$

$$= \min(1, \max(a-\eta, b+\varepsilon)) - \max(0, \min(b+\varepsilon-\eta, a-\eta)) \geq 0,$$

because for $p, q \in [0,1]$:

$$\min(1, \max(p,q)) \geq \max(0, \min(p,q)).$$

If $b+\varepsilon \geq 1$, then

$$X = \min(1, b+\varepsilon) - \max(0, \min(\max(0, 1-\eta), a-\eta))$$

$$= 1 - \max(0, \min(1-\eta, a-\eta)) = 1 - \max(0, a-\eta) = 1-(a-\eta) \geq 0.$$

Therefore, Property 1 is valid only as an IFT.

(b) is proved analogously.

(c) Let x be a given propositional form. We shall study the relation between the first components of $V(x)$ and $V(\neg^{\varepsilon,\eta}\neg^{\varepsilon,\eta}\neg^{\varepsilon,\eta}x)$.

$$V(\neg^{\varepsilon,\eta}\neg^{\varepsilon,\eta}\neg^{\varepsilon,\eta}x) = \neg^{\varepsilon,\eta}\neg^{\varepsilon,\eta}\neg^{\varepsilon,\eta}\langle a,b\rangle = \neg^{\varepsilon,\eta}\neg^{\varepsilon,\eta}\langle \min(1, b+\varepsilon), \max(0, a-\eta)\rangle$$

$$= \neg^{\varepsilon,\eta}\langle \min(1, \max(\varepsilon, a-\eta+\varepsilon)), \max(0, \min(1-\eta, b+\varepsilon-\eta))\rangle$$

$$= \langle \min(1, \max(\varepsilon, \min(1-\eta+\varepsilon, b+2\varepsilon-\eta))), \max(0, \min(1-\eta, \max(\varepsilon-\eta, a-2\eta+\varepsilon)))\rangle.$$

Let

$$X = \min(1, b+\varepsilon) - \min(1, \max(\varepsilon, \min(1-\eta+\varepsilon, b+2\varepsilon-\eta))).$$

If $b+\varepsilon \geq 1$, then $1-\eta+\varepsilon \leq b+2\varepsilon-\eta$ and

$$X = 1 - \min(1, \max(\varepsilon, 1-\eta+\varepsilon)) \geq 0.$$

If $b+\varepsilon < 1$, then $1-\eta+\varepsilon > b+2\varepsilon-\eta$ and

$$X = b+\varepsilon - \min(1, \max(\varepsilon, b+2\varepsilon-\eta))$$

$$= b+\varepsilon - \max(\varepsilon, b+2\varepsilon-\eta).$$

If $\varepsilon \geq b+2\varepsilon-\eta$, then

$$X = b+\varepsilon-\varepsilon = b \geq 0.$$

If $\varepsilon < b+2\varepsilon-\eta$, then

$$X = b+\varepsilon-b-2\varepsilon+\eta = \eta-\varepsilon > 0.$$

Therefore, for the first componets of $V(x)$ and $V(\neg^{\varepsilon,\eta}\neg^{\varepsilon,\eta}\neg^{\varepsilon,\eta}x)$ the equality is not valid. Now, we shall study the relation between the second componets of $V(x)$ and $V(\neg^{\varepsilon,\eta}\neg^{\varepsilon,\eta}\neg^{\varepsilon,\eta}x)$.

Let

$$Y = \max(0, \min(1-\eta, \max(\varepsilon-\eta, a-2\eta+\varepsilon))) - \max(0, a-\eta).$$

If $a-\eta \leq 0$, then

$$Y = \max(0, \min(1-\eta, \varepsilon-\eta)) = \max(0, \varepsilon-\eta)) = 0.$$

If $a - \eta > 0$, then, having in mind that

$$1 - \eta - a + 2\eta - \varepsilon = 1 - a + \eta - \varepsilon > 1 - a \geq 0$$

we obtain

$$Y = \max(0, \min(1 - \eta, a - 2\eta + \varepsilon)) - a + \eta) = \max(0, a - 2\eta + \varepsilon) - a + \eta) \leq 0.$$

Therefore, the relation between the second componets of $V(x)$ and $V(\neg^{\varepsilon,\eta}\neg^{\varepsilon,\eta}\neg^{\varepsilon,\eta}x)$ does not permit to assert that $V(x) \geq V(\neg^{\varepsilon,\eta}\neg^{\varepsilon,\eta}\neg^{\varepsilon,\eta}x)$.

All following assertions are proved similarly and for this reason we will omit their proofs.

In [9, 10] we classify each couple $(\neg, \rightarrow)$ as:

• classical – it satisfies properties P1, P2, P3 and for each x: $V(x) = V(\neg\neg x)$;

• intuitionistical – it satisfies properties P1, P3 and does not satisfy property P2;

• non-standard – it satisfies some (between zero and three) of properties P1, P2, P3 and there is x: $V(x) \neq V(\neg\neg x)$.

Therefore, the new implication is from non-standard type.

Now, we shall study the validity of the LEM and the De Morgan's Laws in the different forms, described above.

Theorem 2: Negation $\neg^{\varepsilon}$

(a) satisfies the LEM in its IFT-form, but not in its tautological form.

(b) satisfies the Modified LEM in its IFT-form, but not in its tautological form.

Theorem 3: Negation $\neg^{\varepsilon,\eta}$

(a) satisfies the De Morgan's Laws in the forms (1) and (2);

(b) does not satisfy the De Morgan's Laws in the forms (3) and (4);

(c) doen not satisfy the De Morgan's Laws in the forms (5) and (6);

(d) satisfies the De Morgan's Laws in the forms:

$$\neg^{\varepsilon,\eta}(\neg_1 x \vee \neg_1 y) = \neg^{\varepsilon,\eta}\neg^{\varepsilon,\eta}x \wedge \neg^{\varepsilon,\eta}\neg^{\varepsilon,\eta}y,$$

$$\neg^{\varepsilon,\eta}(\neg_1 x \wedge \neg_1 y) = \neg^{\varepsilon,\eta}\neg^{\varepsilon,\eta}x \vee \neg^{\varepsilon,\eta}\neg^{\varepsilon,\eta}y.$$

Finally, we shall check the Georg Klir and Bo Yuan's axioms for implication (see [14]):

Axiom 1 $(\forall x, y)(x \leq y \rightarrow (\forall z)(I(x, z) \geq I(y, z))$.

Axiom 2 $(\forall x, y)(x \leq y \rightarrow (\forall z)(I(z, x) \leq I(z, y))$.

Axiom 3 $(\forall y)(I(0, y) = 1)$.

Axiom 4 $(\forall y)(I(1, y) = y)$.

Axiom 5 $(\forall x)(I(x, x) = 1)$.

Axiom 6 $(\forall x, y, z)(I(x, I(y, z)) = I(y, I(x, z)))$.

Axiom 7 $(\forall x, y)(I(x, y) = 1 \text{ iff } x \leq y)$.

Axiom 8 $(\forall x, y)(I(x, y) = I(N(y), N(x)))$, where N is an operation for a negation.

Axiom 9 I is a continuous function.

Theorem 4: Implication $\rightarrow^{\varepsilon}$ and negation $\neg^{\varepsilon,\eta}$

(a) satisfy Axioms 1,2,3,6 and 9;

(b) satisfy Axioms 4 and 5 as IFTs, but not as tautologies;

(c) satisfy Axiom 8 in the form

Axiom 8' $(\forall x, y)(I(x, y) \leq I(N(y), N(x)))$.

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