Seventh Int. Conf. on IFSs, Sofia, 23-24 August 2003
NIFS Vol. 9 (2003), 3, 51-59

A Concept of Similarity for Intuitionistic Fuzzy Sets and its Use in the Aggregation of Experts' Testimonies

Eulalia Szmidt and Janusz Kacprzyk

Systems Research Institute, Polish Academy of Sciences
ul. Newelska 6, 01–447 Warsaw, Poland
E-mail: {szmidt, kacprzyk}@ibspan.waw.pl

Abstract

In this article we apply a new measure of similarity to analyse the extent of agreement in a group of experts. The proposed measure takes into account not only a pure distance between intuitionistic fuzzy preferences but also examines if the compared preferences are more similar or more dissimilar. The agreement of a whole group is assessed via an aggregation of individual testimonies expressed by intuitionistic fuzzy preference relations.
Keywords: intuitionistic fuzzy sets, distances, similarity measure, group agreement.

1 Introduction

The nature of similarity is broadly explored and discussed in the recent book by Cross and Sudkamp [2]. They stressed the fundamental role of compatibility and similarity in inference and in applications in approximate reasoning using fuzzy set theory. The analysis of the similarity is as well a fundamental task when employing intuitionistic fuzzy sets witch are generalization of fuzzy sets (Atanassov [1]).

In this article we propose and apply a new measure of similarity to compare intuitionistic fuzzy preferences given by individuals (experts) and to assess an extent of a group agreement.

The similarity measure we introduce is not a standard similarity measure in the sense that it is not a dual concept to a distance (Tversky [17]). In commonly used similarity measures dissimilarity behaves like a distance function. Such a standard approach - formulated for objects as crisp values was later extended and used to assess similarity of fuzzy sets (Cross and Sudkamp [2]). Distances were also proposed to measure similarity between intuitionistic fuzzy sets (Dengfeng and Chuntian [3]). The measure we propose is not that kind of similarity - it does not measure just a distance between the compared intuitionistic fuzzy preferences given by individuals. The new measure answers the question if the compared preferences are more similar or more dissimilar. So in fact the name "similarity measure" is not perfect in the classical sense - the objects are identical when the measure is equal to zero, are to the same extent similar as dissimilar when the measure is equal to 1, and are absolutely dissimilar when the measure approaches ∞. So the name "dissimilarity" would be also not correct in the classical sense.

In Szmidt [5], Szmidt and Kacprzyk [7, 8, 10, 13, 16], Kacprzyk and Szmidt [4] we use intuitionistic fuzzy sets to solve group decision problems, and to determine soft measures of consensus.

We have a set of n options, $S = \{s_1, \ldots, s_n\}$, and a set of m individuals, $I = \{1, \ldots, m\}$. In the classic fuzzy approach, each individual k provides his or her individual fuzzy preference relation, R_k, given by $\mu_{R_k} : S \times S \rightarrow [0, 1]$. An individual fuzzy preference relation may be represented by a matrix $[r_{ij}^k]$ such that $r_{ij}^k = \mu_{R_k}(s_i, s_j)$; $i, j = 1, \ldots, n$; $k = 1, \ldots, m$; $[r_{ij}^k] + [r_{ji}^k] = 1$.

Here, we use intuitionistic fuzzy preference relations. Each individual k provides his or her *(individual) intuitionistic fuzzy preference relation*, giving not only R_k (given, as previously, by its membership function μ_{R_k}) but also Π_k – so-called intuitionistic fuzzy index, $\pi_k : S \times S \rightarrow [0, 1]$, conveniently represented by a matrix $[\pi_{ij}^k(s_i, s_j)]$; $i, j = 1, \ldots, n$; $k = 1, \ldots, m$. Such a representation of individual preferences, with an added intuitionistic fuzzy index, can better reflect the very imprecision of testimonies (expressing individual preferences).

The organization of the paper is as follow. First, intuitionistic fuzzy sets (Atanassov, [1]) are presented in a brief way. Next, a concept of distances between intuitionistic fuzzy sets is reminded and the method of analyzing similarity is described. Finally, we give an example illustrating an application of the proposed measure - an extent of agreement in a group of experts is examined.

2 Brief introduction to intuitionistic fuzzy sets

As opposed to a fuzzy set in X(Zadeh [18]) , given by

$$A' = \{< x, \mu_{A'}(x) > | x \in X\} \tag{1}$$

where $\mu_{A'}(x) \in [0, 1]$ is the membership function of the fuzzy set A', an intuitionistic fuzzy set (Atanassov [1]) A is given by

$$A = \{< x, \mu_A(x), \nu_A(x) > | x \in X\} \tag{2}$$

where: $\mu_A : X \rightarrow [0, 1]$ and $\nu_A : X \rightarrow [0, 1]$ such that

$$0 \leq \mu_A(x) + \nu_A(x) \leq 1 \tag{3}$$

and $\mu_A(x)$, $\nu_A(x) \in [0, 1]$ denote a degree of membership and a degree of non-membership of $x \in A$, respectively.

Obviously, each fuzzy set may be represented by the following intuitionistic fuzzy set

$$A = \{< x, \mu_{A'}(x), 1 - \mu_{A'}(x) > | x \in X\} \tag{4}$$

For each intuitionistic fuzzy set in X, we will call

$$\pi_A(x) = 1 - \mu_A(x) - \nu_A(x) \tag{5}$$

an *intuitionistic fuzzy index* (or a *hesitation margin*) of $x \in A$ and, it expresses a lack of knowledge of whether x belongs to A or not (cf. Atanassov [1]). It is obvious that $0 \leq \pi_A(x) \leq 1$, for each $x \in X$.

In our further considerations we will use the notion of the complement elements, which definition is a simple consequence of a complement set A^C

$$A^C = \{< x, \nu_A(x), \mu_A(x) > | x \in X\} \tag{6}$$

2.1 Distances between intuitionistic fuzzy sets

In [5, 6, 12] it is shown why when calculating distances between intuitionistic fuzzy sets it is necessary to take into account all three parameters describing intuitionistic fuzzy sets. One of the reasons is that when taking into account two parameters only, for elements from classical fuzzy sets (which are a special case of intuitionistic fuzzy sets) we obtain distances from a different interval than for elements belonging to intuitionistic fuzzy sets. It practically makes it impossible to consider by the same formula the two types of sets. For more details we refer the interested reader to [5, 6, 12].

In our further considerations we will use the normalized Hamming distance between intuitionistic fuzzy sets A, B in $X = \{x_1, x_2, \ldots, x_n\}$ [5, 6, 12]:

$$\begin{aligned} l_{IFS}(A, B) &= \frac{1}{2n} \sum_{i=1}^{n} (|\mu_A(x_i) - \mu_B(x_i)| + |\nu_A(x_i) - \nu_B(x_i)| + \\ &+ |\pi_A(x_i) - \pi_B(x_i)|). \end{aligned} \tag{7}$$

For (7) we have:

$$0 \leq l_{IFS}(A, B) \leq 1. \tag{8}$$

3 Similarity measure

We propose here a new similarity measure for intuitionistic fuzzy sets. We use a geometrical interpretation of intuitionistic fuzzy sets which was described in details by Szmidt ([5]), Szmidt and Baldwin ([6]), Szmidt and Kacprzyk ([12],[14]). Here we remind only that each element belonging to an intuitionistic fuzzy set can be represented as a point (μ, ν, π) belonging to the triangle ABD (Figure **??**). Point A represents elements fully belonging to an intuitionistic fuzzy set ($\mu = 1$), point B represents elements fully not belonging to an intuitionistic fuzzy set ($\nu = 1$), point D represents elements with hesitation margin $\pi = 1$ i.e, about which we are not able to say if they belong or not belong to an intuitionistic fuzzy set. Any other combination of the parameters characteristic for elements belonging to an intuitionistic fuzzy set can be represented inside triangle ABD.

In the simplest situations we assess similarity of any two elements X and F belonging to an intuitionistic fuzzy set (or sets). The proposed measure says if X is more similar to F or to F^C, where F^C is a complement of F. In other words, the proposed measure answers the question if X is more similar or more dissimilar to F.

Definition 1

$$Sim(X,F) = \frac{l_{IFS}(X,F)}{l_{IFS}(X,F^C)} = \frac{a}{b} \tag{9}$$

where: a is a distance(X,F) *from* $X(\mu_X,\nu_X,\pi_X)$ *to* $F(\mu_F,\nu_F,\pi_F)$,
b is the distance(X,F^C) *from* $X(\mu_X,\nu_X,\pi_X)$ *to* $F^C(\nu_F,\mu_F,\pi_F)$,
F^C is a complement of F,
the distances $l_{IFS}(X,F)$ and $l_{IFS}(X,F^C)$ are calculated from (7).

For (9) we have

$$0 \le Sim(X,F) \le \infty \tag{10}$$

and

$$Sim(X,F) = Sim(F,X)$$

(9) can be stated as well as

$$Sim(X,F) = \frac{l_{IFS}(X,F)}{l_{IFS}(X,F^C)} = \frac{l_{IFS}(X^C,F^C)}{l_{IFS}(X,F^C)} = \frac{l_{IFS}(X,F)}{l_{IFS}(X^C,F)} = \frac{l_{IFS}(X^C,F^C)}{l_{IFS}(X^C,F)}$$

It is worth noticing that

- $Sim(X,F) = 0$ means identity of X and F.
- $Sim(X,F) = 1$ means that X is to the same extent similar to F and F^C (i.e., values bigger than 1 mean in fact closer similarity of X and F^C to X and F).
- When $X = F^C$ (or $X^C = F$), i.e. $l_{IFS}(X,F^C)$=$l_{IFS}(X^C,F)$= 0 means complete dissimilarity of X and F (or in other words, identity of X and F^C), and then $Sim(X,F) \to \infty$.
- When $X = F = F^C$ means the highest possible entropy (see [14]) for both elements F and X i.e. the highest "fuzziness" – not too constructive case when looking for compatibility (both similarity and dissimilarity).

In other words, when applying the measure (9) to analyse the similarity of two objects, one should be interested in the values $0 \le Sim(X,F) < 1$.

In Szmidt and Baldwin ([6]) it is shown that the proposed measure of of similarity (9) between $X(\mu_X,\nu_X,\pi_X)$ and $F(\mu_F,\nu_F,\pi_F)$ is more powerful then a simple distance between them.

4 Analysis of agreement in a group of experts

We will use the new concept of similarity to analyse the extent of agreement between experts i.e., to say if all of the considered pairs of expert's preferences are

- just the same (i.e. full agreement meaning consensus in a traditional sense - the proposed measure of similarity is equal to 0),
- quite opposite (i.e. full disagreement - similarity is equal to infinity),
- different to some extent (what means that a distance from consensus is from the open interval $(0, 1)$)
- to the same extent similar as dissimilar - the proposed measure of similarity is equal to 1.

Preferences given by each individual are expressed via intuitionistic fuzzy sets (describing intuitionistic fuzzy preferences). Having in mind that distances between intuitionistic fuzzy sets must be calculated with taking into account all three parameters characterizing an intuitionistic fuzzy set, we start from a set of data which consists of three types of matrices describing individual preferences. The first type of matrices is the same as for classical fuzzy sets, i.e. membership functions $[r_{ij}^k]$ given by each individual k concerning each pair of options ij. But, additionally, it is necessary to take into account hesitation margins $[\pi_{ij}^k]$ and non-membership functions $[\nu_{ij}^k]$.

In general, the extent of similarity for two experts k_1, k_2 considering n options is given as

$$\begin{aligned} Sim^{k_1,k_2} &= \frac{1}{A}\sum_{i=1}^{n-1}\sum_{j=i+1}^{n} Sim^{k_1,k_2}(i,j) = \frac{1}{A}[\sum_{i=1}^{n-1}\sum_{j=i+1}^{n}(\mid \mu_{ij}(k_1) - \mu_{ij}(k_2) \mid + \mid \nu_{ij}(k_1) - \nu_{ij}(k_2) \mid + \\ &+ \mid \pi_{ij}(k_1) - \pi_{ij}(k_2) \mid)]/[\sum_{i=1}^{n-1}\sum_{j=i+1}^{n}(\mid \mu_{ij}(k_1) - \nu_{ij}(k_2) \mid + \mid \nu_{ij}(k_1) - \mu_{ij}(k_2) \mid + \\ &+ \mid \pi_{ij}(k_1) - \pi_{ij}(k_2) \mid)] \end{aligned} \tag{11}$$

where

$$A = \frac{1}{2C_n^2} = \frac{1}{n(n-1)} \tag{12}$$

When we have m experts, we examine similarity of their preferences in pairs (11) and next, we find an agreement of all experts

$$Sim = \frac{1}{m(m-1)}\sum_{p=1}^{m-1}\sum_{r=p+1}^{m} Sim^{k_p,k_r} \tag{13}$$

where Sim^{k_p,k_r} is given by (11).

Example 1 Suppose that there are 3 individuals ($m = 3$) considering 3 options ($n = 3$), and the individual intuitionistic fuzzy preference relations are:

$$\mu^1(i,j) = \begin{bmatrix} - & .1 & .5 \\ .9 & - & .5 \\ .4 & .3 & - \end{bmatrix} \nu^1(i,j) = \begin{bmatrix} - & .9 & .4 \\ .1 & - & .3 \\ .5 & .5 & - \end{bmatrix} \pi^1(i,j) = \begin{bmatrix} - & 0 & .1 \\ 0 & - & .2 \\ .1 & .2 & - \end{bmatrix}$$

$$\mu^2(i,j) = \begin{bmatrix} - & .1 & .5 \\ .9 & - & .5 \\ .2 & .2 & - \end{bmatrix} \nu^2(i,j) = \begin{bmatrix} - & .9 & .2 \\ .1 & - & .2 \\ .5 & .5 & - \end{bmatrix} \pi^2(i,j) = \begin{bmatrix} - & 0 & .3 \\ 0 & - & .3 \\ .3 & .3 & - \end{bmatrix}$$

$$\mu^3(i,j) = \begin{bmatrix} - & .2 & .1 \\ .8 & - & .6 \\ .2 & .3 & - \end{bmatrix} \nu^3(i,j) = \begin{bmatrix} - & .8 & .2 \\ .2 & - & .3 \\ .1 & .6 & - \end{bmatrix} \pi^3(i,j) = \begin{bmatrix} - & 0 & .7 \\ 0 & - & .1 \\ .7 & .1 & - \end{bmatrix}$$

To find out the extent of agreement in the group, we must calculate similarity $Sim^{p,r}(i,j)$ for each pair of experts (p,r) considering each pair of options (i,j).

First, we calculate similarity for each pair of experts concerning the first and the second option. For example, the data and the calculations for the second and the third experts are
$F^2(1,2) = (0.1, 0.9, 0)$ - preferences of the second expert,
$F^3(1,2) = (0.2, 0.8, 0)$ - preferences of the third expert,
$F^{3,C}(1,2) = (0.8, 0.2, 0)$ - the complement of $F^3(1,2)$, i.e., opposite preferences of the third expert.

¿From (9) and (11) we have

$$Sim^{2,3}(1,2) = \frac{l(F^2(1,2), F^3(1,2))}{l(F^2(1,2), F^{3,C}(1,2))} = \frac{0.1}{0.7} = 0.14 \tag{14}$$

Similar calculations for experts (1,2) and (1,3) give respectively

$$Sim^{1,2}(1,2) = \frac{l(F^1(1,2), F^2(1,2))}{l(F^1(1,2), F^{2,C}(1,2))} = 0 \tag{15}$$

$$Sim^{1,3}(1,2) = \frac{l(F^1(1,2), F^3(1,2))}{l(F^1(1,2), F^{3,C}(1,2))} = \frac{0.1}{0.7} = 0.2 \tag{16}$$

¿From (14)-(16) we obtain an average similarity for the three considered experts considering options $(1,2)$, namely

$$Sim(1,2) = \frac{1}{3}(0 + 0.2 + 0.14) = 0.11 \tag{17}$$

Similar calculations for options $(1,3)$ give following results

$$Sim^{1,2}(1,3) = \frac{l(F^1(1,3), F^2(1,3))}{l(F^1(1,3), F^{2,C}(1,3))} = 0.67 \tag{18}$$

$$Sim^{1,3}(1,3) = \frac{l(F^1(1,3), F^3(1,3))}{l(F^1(1,3), F^{3,C}(1,3))} = 1 \quad (19)$$

$$Sim^{2,3}(1,3) = \frac{l(F^2(1,3), F^3(1,3))}{l(F^2(1,3), F^{3,C}(1,3))} = 1 \quad (20)$$

After aggregation the above values we obtain similarity for options $(1,3)$

$$Sim(1,3) = \frac{1}{3}(0.67 + 1 + 1) = 0.89 \quad (21)$$

And, finally for options $(2,3)$ we have

$$Sim^{1,2}(2,3) = \frac{l(F^1(2,3), F^2(2,3))}{l(F^1(2,3), F^{2,C}(2,3))} = 0.33 \quad (22)$$

$$Sim^{1,3}(2,3) = \frac{l(F^1(2,3), F^3(2,3))}{l(F^1(2,3), F^{3,C}(2,3))} = 0.33 \quad (23)$$

$$Sim^{2,3}(2,3) = \frac{l(F^2(2,3), F^3(2,3))}{l(F^2(2,3), F^{3,C}(2,3))} = 0.57 \quad (24)$$

Aggregation of the above values gives similarity for options $(2,3)$

$$Sim(2,3) = \frac{1}{3}(0.33 + 0.33 + 0.57) = 0.41 \quad (25)$$

The above results show that the biggest agreement in our group concerns options $(1,2)$ - the similarity measure is equal to 0.11. The smallest agreement concerns options $(1,3)$ - the similarity measure is equal to 0.89.

Of course, similar calculations can be performed for experts (aggregation is performed by experts). The results are

$$Sim^{1,2} = 0.33 \quad (26)$$

$$Sim^{1,3} = 0.51 \quad (27)$$

$$Sim^{2,3} = 0.57 \quad (28)$$

The preferences of the first and the second expert are the most similar (26), the preferences of the second and the third experts are the least similar (28).

Aggregation of the results (26)-(28) gives the similarity measure aggregated both by options and by experts (the general similarity for the group)

$$Sim = \frac{1}{3}(0.33 + 0.51 + 0.57) = 0.47 \quad (29)$$

Of course, just the same results will be obtained when aggregating (17), (21) and (25).

In our example an agreement of the group of experts (similarity concerning all options) is equal to 0.47 (not bad). □

It is worth noticing that the presented method of a group agreement analysis makes it possible to take into account the fact that some experts can be more important than others - proper weights for pairs of individuals can be taken into account into formula (13).

5 Concluding remarks

We applied a new measure of similarity in the aggregation of experts' testimonies. For intuitionistic fuzzy sets with the additional degrees of freedom - non-memberships, it seems important to use similarity measures taking into account these additional parameters in a way introducing new quality to the process of assessing similarity. It can be achieved by comparing distances to an interesting object/element/preference and its complement.

References

[1] Atanassov K. (1999), Intuitionistic Fuzzy Sets: Theory and Applications. Springer-Verlag.

[2] Cross V. and Sudkamp T. (2002) Similarity and Compatibility in Fuzzy Set Theory. Physica-Verlag.

[3] L. Dengfeng and C. Chuntian (2002) New similarity measures of intuitionistic fuzzy sets and application to pattern recognitions. Pattern Recognition Letters, Vol. 23, pp. 221-225.

[4] Kacprzyk J. and Szmidt E. (2000) On Measures on Consensus Under Intuitionistic Fuzzy Relations. 8th Int. Conf. IPMU, Madrid, July 3-7, 2000, 641-647.

[5] Szmidt E. (2000) Applications of Intuitionistic Fuzzy Sets in Decision Making. (D.Sc. dissertation) Techn. Univ., Sofia, 2000.

[6] Szmidt E. and Baldwin J. (2003) New Similarity Measure for Intuitionistic Fuzzy Set Theory and Mass Assignment Theory. Proc. of the 7-th Int. Conf. on IFSs. Sofia, 2003.

[7] Szmidt E. and Kacprzyk J. (1996a) Intuitionistic fuzzy sets in group decision making, *Notes on IFS*, 2, pp. 15–32.

[8] Szmidt E. and Kacprzyk J. (1996c) Remarks on some applications of intuitionistic fuzzy sets in decision making, *Notes on IFS*, Vol.2, No.3, pp. 22–31.

[9] Szmidt E. and Kacprzyk J. (1997) On measuring distances between intuitionistic fuzzy sets, Notes on IFS, Vol.3, No.4, pp.1 - 13.

[10] Szmidt E. and Kacprzyk J. (1998a) Group Decision Making under Intuitionistic Fuzzy Preference Relations. 7th Int. Conf. IPMU'98 (Paris, La Sorbonne, July 6-10), pp. 172-178.

[11] Szmidt E. and Kacprzyk J. (1998b) Applications of Intuitionistic Fuzzy Sets in Decision Making. 8th Congr. EUSFLAT'99 (Pamplona, September 8-10), pp. 150–158.

[12] Szmidt E. and Kacprzyk J. (2000) Distances between intuitionistic fuzzy sets, Fuzzy Sets and Systems, Vol.114, No.3, pp.505 – 518.

[13] Szmidt E. and Kacprzyk J. (2001) Distance from Consensus Under Intuitionistic Fuzzy Preferences. Proc. EUROFUSE Workshop on Preference Modelling and Applications, Granada, April 25–27, pp. 73–78.

[14] Szmidt E., Kacprzyk J. (2001) Entropy for intuitionistic fuzzy sets. Fuzzy Sets and Systems, vol. 118, No. 3, pp. 467–477.

[15] Szmidt E. and Kacprzyk J. (2001) Analysis of Consensus under Intuitionistic Fuzzy Preferences. Proc. Int. Conf. in Fuzzy Logic and Technology. De Montfort Univ. Leicester, UK, September 5-7, pp. 79–82.

[16] Szmidt E. and Kacprzyk J. (2002) Analysis of Agreement in a Group of Experts via Distances Between Intuitionistic Fuzzy Preferences. Proc. 9th Int. Conf. IPMU 2002, Annecy, France, July 1–5, pp. 1859–1865.

[17] Tversky A. (1977) Features of similarity. Psychol. Rev. Vol. 84, pp. 327–352.

[18] L.A. Zadeh (1965) Fuzzy sets. Information and Control, 8, 338–353.

[19] Zadeh, L.A. (1983). A computational approach to fuzzy quantifiers in natural languages. *Comput. Math. Appl.*, Vol. 9, No. 1, pp. 149–184.