A descriptive definition of the probability on intuitionistic fuzzy sets

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Abstract

In [2] a general probability theory has been constructed for intuitionistic fuzzy events ([1]) defined on any probability space $(\Omega, \mathcal{S}, P)$. To any element A belonging to the family $\mathcal{F}$ of all intuitionistic fuzzy events a compact interval $\mathcal{P}(A)$ on the real line is assigned. In the paper we consider a mapping $\mathcal{P} : \mathcal{F} \to \mathcal{J}$, where $\mathcal{J}$ is the family of all compact intervals. Some properties of $\mathcal{P}$ are postulated axiomatically. Then a representation theorem is proved stating that to any mapping $\mathcal{P}$ satisfying the properties there exists a probability measure $P : \mathcal{S} \to [0,1]$ such that $\mathcal{P}(A)$ can be expressed by the help of P similarly as it has been done in [2].

1 Introduction

Let $(\Omega, \mathcal{S}, P)$ be a probability space. By an intuitionistic fuzzy event (IFE) we understand a couple $A = (\mu_A, \nu_A)$ of nonnegative, $\mathcal{S}$-measurable functions such that $\mu_A + \nu_A \leq 1$. The number $\mu_A(x)$ represents the degree of membership, the number $\nu_A(x)$ the degree of nonmembership. Probability $\mathcal{P}(A)$ is defined as the interval $\mathcal{P}(A) = [p^\flat(A), p^\sharp(A)]$, where

$$p^\flat(A) = \int_\Omega \mu_A dP, \; p^\sharp(A) = 1 - \int_\Omega \nu_A dP. \quad (1)$$

In [2] some properties of $\mathcal{P}$ has been proved, as monotonicity, additivity, and continuity. In this paper we shall prove another type of additivity based on the Lukasiewicz connectives ([3]). In Section 2 we introduce some characteristic properties of the mapping $\mathcal{P}$ and in Section 3 we prove the corresponding representation theorem.

2 Characteristic properties of probability

As in the classical case the properties characterising probability are additivity and continuity. In this paper we shall use additivity based on the Lukasiewicz connectives $\oplus, \odot$. Recall that

$$x \odot y = (x + y - 1) \vee 0$$

$$x \oplus y = (x + y) \wedge 1$$

for any $x, y \in [0,1]$. For elements $A = (\mu_A, \nu_A) \in \mathcal{F}, B = (\mu_B, \nu_B) \in \mathcal{F}$ we define

$$(\mu_A, \nu_A) \odot (\mu_B, \nu_B) = (\mu_A \odot \mu_B, \nu_A \oplus \nu_B)$$

$$(\mu_A, \nu_A) \oplus (\mu_B, \nu_B) = (\mu_A \oplus \mu_B, \nu_A \odot \nu_B).$$

Denote further $\mathbf{0} = (0,1)$, and consider probability $\mathcal{P}$ defined by the formulas (1). Define the operation $+$ on the family $\mathcal{J}$ of all compact intervals by

$$[a,b] + [c,d] = [a + c, b + d]$$

Then we can prove the following proposition.

Proposition 1. *If $A \odot B = \mathbf{O}$, then $\mathcal{P}(A \oplus B) = \mathcal{P}(A) + \mathcal{P}(B)$.*

Proof. Put $A = (\mu_A, \nu_A), B = (\mu_B, \nu_B)$. Then

$$(0,1) = (\mu_A \odot \mu_B, \nu_A \oplus \nu_B) =$$
$$((\mu_A + \mu_B - 1) \vee 0, (\nu_A + \nu_B) \wedge 1).$$

We have obtained

$$0 = (\mu_A + \mu_B - 1) \vee 0,$$

$$1 = (\nu_A + \nu_B) \wedge 1,$$

hence

$$\mu_A + \mu_B \leq 1, \nu_A + \nu_B \geq 1. \qquad (2)$$

Now

$$A \oplus B = (\mu_A \oplus \mu_B, \nu_A \odot \nu_B) =$$

$$= ((\mu_A + \mu_B) \wedge 1, (\nu_A + \nu_B - 1) \vee 0)$$

$$= (\mu_A + \mu_B, \nu_A + \nu_B - 1).$$

Therefore

$$\mathcal{P}(A \oplus B) = [\int_\Omega (\mu_A + \mu_B) dP, 1 - \int_\Omega ((\nu_A + \nu_B) - 1) dP] \qquad (3)$$

$$= [\int_\Omega \mu_A dP + \int_\Omega \mu_B dP, 1 - \int_\Omega \nu_A dP + 1 - \int_\Omega \nu_B dP].$$

On the other hand

$$\mathcal{P}(A) + \mathcal{P}(B) = [\int_\Omega \mu_A dP, 1 - \int_\Omega \nu_A dP] \qquad (4)$$

$$+ [\int_\Omega \mu_B dP, 1 - \int_\Omega \nu_B dP]$$

$$= [\int_\Omega \mu_A dP + \int_\Omega \mu_B dP, 1 - \int_\Omega \nu_A dP + 1 - \int_\Omega \nu_B dP].$$

By (3) and (4) we obtain $\mathcal{P}(A \oplus B) = \mathcal{P}(A) + \mathcal{P}(B)$.

To formulate continuity consider a sequence $(A_n)_{n=1}^\infty$ of IFE and $A \in \mathcal{F}$. Denote $A_n = (\mu_{A_n}, \nu_{A_n}), A = (\mu, \nu)$. We shall write $A_n \nearrow A$, if

$$\mu_{A_n}(\omega) \nearrow \mu_A(\omega), \nu_{A_n} \searrow \nu_A(\omega)$$

for all $\omega \in \Omega$. If $(I_n)_{n=1}^\infty$ is a sequence of compact intervals, $I_n = [a_n, b_n], I = [a, b]$, then we write $I_n \nearrow I$, if $a_n \nearrow a, b_n \nearrow b$. The following theorem holds.

Proposition 2. *If $A_n \nearrow A$, then $\mathcal{P}(A_n) \nearrow \mathcal{P}(A)$.*

Proof. [2], Prop. 5.

Of course, besides the continuity and our variant of additivity there hold many further properties. We shall need the following three ones. First, evidently

$$\mathcal{P}((1_\Omega, 0_\Omega)) = [\int_\Omega 1_\Omega dP, 1 - \int_\Omega 0_\Omega dP] = [1, 1] = \{1\}.$$

Secondly, if we denote$\mathcal{P}(A) = [p^\flat(A), p^\sharp(A)], A = (\mu, \nu)$, then

$$p^\flat(A) = \int_\Omega \mu dP$$

depends only on μ, and

$$p^\sharp(A) = 1 - \int_\Omega \nu dp$$

depends only on ν. Thirdly, if $A \in \mathcal{S}$, and we denote $I(A) = (\chi_A, \chi_{A'})$, then

$$p^\flat(I(A)) = \int_\Omega \chi_A dP = P(A),$$

$$p^\sharp(I(A)) = 1 - \int_\Omega \chi_{A'} dP = 1 - P(A') = P(A),$$

hence

$$p^\flat(I(A)) = p^\sharp(I(A)).$$

3 Representation theorem

Denote by $\mathcal{F}$ the set of all intuitionistic fuzzy events with respect to a given measurable space $(\Omega, \mathcal{S})$, where $\mathcal{S}$ is a σ-algebra of subsets of Ω. Denote by $\mathcal{J}$ the set of all compact inervals on the real line. If $A \in \mathcal{S}$ then we define $I(A) \in \mathcal{F}$ by the formula $I(A) = (\chi_A, \chi_{A'})$. If $\mathcal{P} \to \mathcal{J}$ then we write $\mathcal{P}((\mu, \nu)) = [p^\flat(\mu, \nu), p^\sharp(\mu, \nu)]$.

Theorem. *Let $\mathcal{P} : \mathcal{F} \to \mathcal{J}$ satisfy the following properties:*

$$\mathcal{P}((1_\Omega, 0_\Omega)) = \{1\}; \qquad (i)$$

$$A, B \in \mathcal{F}, A \odot B = (0_\Omega, 1_\Omega) \implies$$

$$\mathcal{P}(A \oplus B) = \mathcal{P}(A) + \mathcal{P}(B); \qquad (ii)$$

$$A_n \in \mathcal{F}(n = 1, 2, ...), A \in \mathcal{F}, A_n \nearrow A \implies$$

$$\mathcal{P}(A_n) \nearrow \mathcal{P}(A); \qquad (iii)$$

$$(\mu, \nu), (\mu, \kappa), (\lambda, \nu) \in \mathcal{F} \implies$$

$$p^\flat(\mu, \nu) = p^\flat(\mu, \kappa), p^\sharp(\mu, \nu) = p^\sharp(\lambda, \nu); \qquad (iv)$$

$$\forall a \in \mathcal{S} : \; p^\flat(I(A)) = p^\sharp(I(A)). \qquad (v)$$

Then there exists a probability measure $P : \mathcal{S} \to [0,1]$ such that

$$\mathcal{P}((\mu,\nu)) = [\int_\Omega \mu dP, 1 - \int_\Omega \nu dP]$$

for any $(\mu,\nu) \in \mathcal{F})$.

Proof. Define $P : \mathcal{S} \to [0,1]$ by the formula $P(A) = p^\flat(I(A)) = p^\flat((\chi_A, \chi_{A'}))$. Evidently, by (i)

$$P(\Omega) = p^\flat((\chi_\Omega, \chi_{\Omega'})) = p^\flat((1_\Omega, 0_\Omega)) = 1.$$

Let $A, B \in \mathcal{S}, A \cap B = \emptyset$. We shall show that

$$I(A) \odot I(B) = (0_\Omega, 1_\Omega) \qquad (5)$$

Indeed

$$I(A) \odot I(B) = (\chi_A, \chi_{A'}) \odot (\chi_B, \chi_{B'}) = (\chi_A \odot \chi_B, \chi_{A'} \oplus \chi_{B'})$$

$$= (\chi_{A\cap B}, \chi_{(A\cap B)'}) = (0_\Omega, 1_\Omega).$$

Similarly

$$I(A) \oplus I(B) = I(A \cup B). \qquad (6)$$

Therefore

$$[p^\flat(I(A\cup B)), p^\sharp(I(A\cup B))] = \mathcal{P}(I(A\cup B))$$
$$= \mathcal{P}(I(A) \oplus I(B)) = \mathcal{P}(I(A)) + \mathcal{P}(I(B))$$
$$= [p^\flat(I(A)), p^\sharp(I(A))] + [p^\flat(I(B)), p^\sharp(IB))]$$
$$= [p^\flat(I(A)) + p^\flat(I(B)), p^\sharp(I(A)) + p^\sharp(I(B))],$$

hence

$$P(A\cup B) = p^\flat(I(A\cup B)) = p^\flat(I(A)) + p^\flat(I(B)) = P(A) + P(B).$$

If $A_n \in \mathcal{S}, A \in S, A_n \nearrow A$, then $\mathcal{P}(A_n) \nearrow \mathcal{P}(A)$, and hence

$$P(A_n) = p^\flat(A_n) \nearrow p^\flat(A) = P(A).$$

We have proved that P is a probability measure. Next step is the proof of the equality

$$p^\flat((\mu, 1-\mu)) = \int_\Omega \mu dP = p^\sharp(\mu, 1-\mu)), \qquad (7)$$

for any $\mathcal{S}$-measurable μ, $\mu : \Omega \to [0,1]$. First let $\mu = \frac{1}{n}\chi_A, A \in \mathcal{S}$. Put $M = (\mu, 1-\mu)$. Then

$$\mathcal{P}(I(A)) = \mathcal{P}(M \oplus M \oplus ... \oplus M)$$
$$= \mathcal{P}(M) + \mathcal{P}(M) + ... + \mathcal{P}(M),$$
$$[p^\flat(I(A)), p^\sharp(I(A))] = n\mathcal{P}(M) = n[p^\flat(M), p^\sharp(M)],$$

hence

$$p^\flat(M) = \frac{1}{n}p^\flat(I(A)) = \frac{1}{n}P(A) = \int_\Omega \frac{1}{n}\chi_A dP.$$

We see that (7) holds for $\mu = \frac{1}{n}\chi_A$. Using again additivity

$$\mathcal{P}((\frac{m}{n}\chi_A, 1 - \frac{m}{n}\chi_A)) =$$

$$= \mathcal{P}((\frac{1}{n}\chi_A, 1 - \frac{1}{n}\chi_A)) + ... + \mathcal{P}((\frac{1}{n}\chi_A, 1 - \frac{1}{n}\chi_A))$$
$$= \int_\Omega \frac{1}{n}\chi_A dP + ... + \int_\Omega \frac{1}{n}\chi_A dP$$
$$= \int_\Omega \frac{m}{n}\chi_A dP,$$

hence (7) holds for $\mu = \frac{m}{n}\chi_A$. Continuity (iii) implies (7) for $\mu = \alpha\chi_A, \alpha \in R$. To prove (7) for any simple $\mathcal{S}$-measurable function $\mu = \sum_{i=1}^n \alpha_i \chi_{A_i}$($A_i$ disjoint, $\alpha_i \in [0,1]$) we use again additivity:

$$\mathcal{P}((\sum_{i=1}^n \alpha_i \chi_{A_i}, 1 - \sum_{i=1}^n \alpha_i \chi_{A_i}))$$

$$= \sum_{i=1}^n \mathcal{P}((\alpha_i \chi_{A_i}, 1 - \alpha_i \chi_{A_i})),$$

$$p^\flat(\sum_{i=1}^n \alpha_i \chi_{A_i}, 1 - \sum_{i=1}^n \alpha_i \chi_{A_i})$$

$$= \sum_{i=1}^n \int_\Omega \alpha_i \chi_{A_i} dP = \int_\Omega (\sum_{i=1}^n \alpha_i \chi_{A_i}) dP$$

Let $(\mu, 1-\mu) \in \mathcal{F}$. Choose μ_n simple such that $0 \le \mu_n \nearrow \mu$. Then $(\mu_n, 1-\mu_n) \nearrow (\mu, 1-\mu)$, hence by (iii)

$$\mathcal{P}((\mu_n, 1-\mu_n)) \nearrow \mathcal{P}((\mu, 1-\mu)),$$

$$p^\flat(\mu, 1-\mu) = lim_{n\to\infty} p^\flat(\mu_n, 1-\mu_n)$$
$$= lim_{n\to\infty} \int_\Omega \mu_n dP = \int_\Omega \mu dP,$$
$$p^\sharp(\mu, 1-\mu) = lim_{n\to\infty} p^\sharp(\mu_n, 1-\mu_n)$$
$$= lim_{n\to\infty} \int_\Omega \mu_n dP = \int_\Omega \mu dP.$$

We have proved (7) for any $\mathcal{S}$-measurable $\mu, 0 \leq \mu \leq 1$. By (iv) we obtain

$$p^\flat(\mu,\nu) = p^\flat(\mu, 1-\mu) = \int_\Omega \mu dP. \qquad (8)$$

On the other hand, since

$$p^\sharp(\mu, 1-\mu) = \int_\Omega \mu dP,$$

putting $\nu = 1-\mu$ we obtain

$$p^\sharp(1-\nu,\nu) = \int_\Omega (1-\nu)dP = 1 - \int_\Omega \nu dP,$$

and again by (iv)

$$p^\sharp(\mu,\nu) = 1 - \int_\Omega \nu dP. \qquad (9)$$

(8) and (9) imply

$$\mathcal{P}((\mu,\nu)) = [p^\flat((\mu,\nu)), p^\sharp((\mu,\nu))]$$
$$= [\int_\Omega \mu dP, 1 - \int_\Omega \nu dP].$$

4 Conclusion

We have shown a possibility to define probability on intuitionistic fuzzy events axiomatically. Therefore it could be possible to develop probability theory only on the base of additivity and continuity.

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5 References

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