Notes on Intuitionistic Fuzzy Sets
Vol. 18, 2012, No. 2, 1–12

A property of the intuitionistic fuzzy implications

Krassimir Atanassov

Dept. of Bioinformatics and Mathematical Modelling
Institute of Biophysics and Biomedical Engineering, Bulgarian Academy of Sciences
105 Acad. G. Bonchev Str., 1113 Sofia, Bulgaria
e-mail: krat@bas.bg

Abstract: It is checked which intuitionistic fuzzy implications satisfy the equality

$$x \to y = x \to (x \to y).$$

Keywords: Implication, Intuitionistic fuzzy set, Operation.
AMS Classification: 03E72.

In memory of Prof. Da Ruan

1 Introduction

The present paper is inspired by Da Ruan's paper [6], in which the logical equality

$$x \to y = x \to (x \to y) \tag{1}$$

is discussed. This equality in some forms will be discussed here for each of the intuitionistic fuzzy implications.

Following [1], the set

$$A^* = \{\langle x, \mu_A(x), \nu_A(x)\rangle | x \in E\}$$

is called an *Intuitionistic Fuzzy Set (IFS),* where the functions $\mu_A : E \to [0,1]$ and $\nu_A : E \to [0,1]$ stand for the degrees of membership and non-membership of the element x from a fixed universe E to the set $A \subset E$, respectively, and every x satisfies that: $0 \le \mu_A(x) + \nu_A(x) \le 1$.

Let for every $x \in E$, the degree of uncertainty have the form

$$\pi_A(x) = 1 - \mu_A(x) - \nu_A(x).$$

Therefore, function π determines the degree of uncertainty.

Let us define the *unit IFS* by:

$$E^* = \{\langle x, 1, 0\rangle | x \in E\}.$$

An IFS A is called *Intuitionistic Fuzzy Tautological Set (IFTS)* if and only if for every $x \in E$

$$\mu_A(x) \geq \nu_A(x).$$

In a series of papers, 138 different intuitionistic fuzzy implications have been introduced. All they are collected in [3]. There, some of their properties are studied, but the equality (1) was not discussed, because the author understood about paper [6] after publishing of [3]. Meanwhile, another implication was constructed in [2], that will be added to the list of the IF-implications in the next Section.

2 Main results

Initially, we give the list of all intuitionistic fuzzy implications (see Table 1). The 139-th implication ($\rightarrow_{139}$) in it, is introduced in [2] and it is not included in [3]. The first of these implications ($\rightarrow_1$) is analogous to Zadeh's fuzzy implication (see, e.g. [4, 5] and by this reason, in [2], it was called "First Zadeh's intuitionistic fuzzy implication", while $\rightarrow_{139}$ obtained the name "Second Zadeh's intuitionistic fuzzy implication".

In the previous publications, containing Table 1, there were some misprints in the formulas in the table, that are corrected here (and in [3]).

Table 1: List of the intuitionistic fuzzy implications

$\rightarrow_1$	$\{\langle x, \max(\nu_A(x), \min(\mu_A(x), \mu_B(x))), \min(\mu_A(x), \nu_B(x))\rangle \| x \in E\}$
$\rightarrow_2$	$\{\langle x, \overline{\text{sg}}(\mu_A(x) - \mu_B(x)), \nu_B(x).\text{sg}(\mu_A(x) - \mu_B(x))\rangle \| x \in E\}$
$\rightarrow_3$	$\{\langle x, 1 - (1 - \mu_B(x)).\text{sg}(\mu_A(x) - \mu_B(x)))),$ $\nu_B(x).\text{sg}(\mu_A(x) - \mu_B(x))\rangle \| x \in E\}$
$\rightarrow_4$	$\{\langle x, \max(\nu_A(x), \mu_B(x)), \min(\mu_A(x), \nu_B(x))\rangle \| x \in E\}$
$\rightarrow_5$	$\{\langle x, \min(1, \nu_A(x) + \mu_B(x)), \max(0, \mu_A(x) + \nu_B(x) - 1)\rangle \| x \in E\}$
$\rightarrow_6$	$\{\langle x, \nu_A(x) + \mu_A(x)\mu_B(x), \mu_A(x)\nu_B(x)\rangle \| x \in E\}$
$\rightarrow_7$	$\{\langle x, \min(\max(\nu_A(x), \mu_B(x)), \max(\mu_A(x), \nu_A(x)),$ $\max(\mu_B(x), \nu_B(x))),\ \max(\min(\mu_A(x), \nu_B(x)),$ $\min(\mu_A(x), \nu_A(x)), \min(\mu_B(x), \nu_B(x)))\rangle \| x \in E\}$
$\rightarrow_8$	$\{\langle x, 1 - (1 - \min(\nu_A(x), \mu_B(x))).\text{sg}(\mu_A(x) - \mu_B(x)),$ $\max(\mu_A(x), \nu_B(x)).\text{sg}(\mu_A(x) - \mu_B(x)),$ $\text{sg}(\nu_B(x) - \nu_A(x))\rangle \| x \in E\}$
$\rightarrow_9$	$\{\langle x, \nu_A(x) + \mu_A(x)^2\mu_B(x), \mu_A(x)\nu_A(x) + \mu_A(x)^2\nu_B(x)\rangle \| x \in E\}$
$\rightarrow_{10}$	$\{\langle x, \mu_B(x).\overline{\text{sg}}(1 - \mu_A(x))$ $+\text{sg}(1 - \mu_A(x)).(\overline{\text{sg}}(1 - \mu_B(x)) + \nu_A(x).\text{sg}(1 - \mu_B(x))),$ $\nu_B(x).\overline{\text{sg}}(1 - \mu_A(x)) + \mu_A(x).\text{sg}(1 - \mu_A(x)).\text{sg}(1 - \mu_B(x))\rangle \| x \in E\}$
$\rightarrow_{11}$	$\{\langle x, 1 - (1 - \mu_B(x)).\text{sg}(\mu_A(x) - \mu_B(x)),$ $\nu_B(x).\text{sg}(\mu_A(x) - \mu_B(x)).\text{sg}(\nu_B(x) - \nu_A(x))\rangle \| x \in E\}$
$\rightarrow_{12}$	$\{\langle x, \max(\nu_A(x), \mu_B(x)), 1 - \max(\nu_A(x), \mu_B(x))\rangle \| x \in E\}$
$\rightarrow_{13}$	$\{\langle x, \nu_A(x) + \mu_B(x) - \nu_A(x).\mu_B(x), \mu_A(x).\nu_B(x)\rangle \| x \in E\}$

$\rightarrow_{14}$	$\{\langle x, 1-(1-\mu_B(x)).\mathrm{sg}(\mu_A(x)-\mu_B(x))$ $-\nu_B(x).\overline{\mathrm{sg}}(\mu_A(x)-\mu_B(x)).\mathrm{sg}(\nu_B(x)-\nu_A(x)),$ $\nu_B(x).\mathrm{sg}(\nu_B(x)-\nu_A(x))\rangle \| x \in E\}$
$\rightarrow_{15}$	$\{\langle x, 1-(1-\min(\nu_A(x),\mu_B(x)))$ $.\mathrm{sg}(\mathrm{sg}(\mu_A(x))-\mu_B(x))+\mathrm{sg}(\nu_B(x))-\nu_A(x)))$ $\min(\nu_A(x)),\mu_B(x)).\mathrm{sg}(\mu_A(x)-\mu_B(x)).\mathrm{sg}(\nu_B(x)-\nu_A(x)),$ $1-(1-\max(\mu_A(x),\nu_B(x))).\mathrm{sg}(\overline{\mathrm{sg}}(\mu_A(x)-\mu_B(x))$ $+\overline{\mathrm{sg}}(\nu_B(x)-\nu_A(x)))-\max(\mu_A(x),\nu_B(x)).\overline{\mathrm{sg}}(\mu_A(x)-\mu_B(x))$ $.\overline{\mathrm{sg}}(\nu_B(x)-\nu_A(x))\rangle \| x \in E\}$
$\rightarrow_{16}$	$\{\langle x, \max(\overline{\mathrm{sg}}(\mu_A(x)),\mu_B(x)), \min(\mathrm{sg}(\mu_A(x)),\nu_B(x))\rangle \| x \in E\}$
$\rightarrow_{17}$	$\{\langle x, \max(\nu_A(x),\mu_B(x)), \min(\mu_A(x).\nu_A(x)+\mu_A(x)^2,\nu_B(x))\rangle \| x \in E\}$
$\rightarrow_{18}$	$\{\langle x, \max(\nu_A(x),\mu_B(x)), \min(1-\nu_A(x),\nu_B(x))\rangle \| x \in E\}$
$\rightarrow_{19}$	$\{\langle x, \max(1-\mathrm{sg}(\mathrm{sg}(\mu_A(x))+\mathrm{sg}(1-\nu_A(x))),\mu_B(x)),$ $\min(\mathrm{sg}(1-\nu_A(x)),\nu_B(x))\rangle \| x \in E\}$
$\rightarrow_{20}$	$\{\langle x, \max(\overline{\mathrm{sg}}(\mu_A(x)),\mathrm{sg}(\mu_B(x))), \min(\mathrm{sg}(\mu_A(x)),\overline{\mathrm{sg}}(\mu_B(x)))\rangle$ $\| x \in E\}$
$\rightarrow_{21}$	$\{\langle x, \max(\nu_A(x),\mu_B(x).(\mu_B(x)+\nu_B(x))),$ $\min(\mu_A(x).(\mu_A(x)+\nu_A(x)),\nu_B(x).$ $(\mu_B(x)^2+\nu_B(x)+\mu_B(x).\nu_B(x)))\rangle \| x \in E\}$
$\rightarrow_{22}$	$\{\langle x, \max(\nu_A(x),1-\nu_B(x)), \min(1-\nu_A(x),\nu_B(x))\rangle \| x \in E\}$
$\rightarrow_{23}$	$\{\langle x, 1-\min(\mathrm{sg}(1-\nu_A(x)),\overline{\mathrm{sg}}(1-\nu_B(x))),$ $\min(\mathrm{sg}(1-\nu_A(x)),\overline{\mathrm{sg}}(1-\nu_B(x)))\rangle \| x \in E\}$
$\rightarrow_{24}$	$\{\langle x, \overline{\mathrm{sg}}(\mu_A(x)-\mu_B(x)).\overline{\mathrm{sg}}(\nu_B(x)-\nu_A(x)),$ $\mathrm{sg}(\mu_A(x)-\mu_B(x)).\mathrm{sg}(\nu_B(x)-\nu_A(x))\rangle \| x \in E\}$
$\rightarrow_{25}$	$\{\langle x, \max(\nu_A(x),\overline{\mathrm{sg}}(\mu_A(x)).\overline{\mathrm{sg}}(1-\nu_A(x)),$ $\mu_B(x).\overline{\mathrm{sg}}(\nu_B(x)).\overline{\mathrm{sg}}(1-\mu_B(x))),$ $\min(\mu_A(x),\nu_B(x))\rangle \| x \in E\}$
$\rightarrow_{26}$	$\{\langle x, \max(\overline{\mathrm{sg}}(1-\nu_A(x)),\mu_B(x)), \min(\mathrm{sg}(\mu_A(x)),\nu_B(x))\rangle \| x \in E\}$
$\rightarrow_{27}$	$\{\langle x, \max(\overline{\mathrm{sg}}(1-\nu_A(x)),\mathrm{sg}(\mu_B(x))),$ $\min(\mathrm{sg}(\mu_A(x)),\overline{\mathrm{sg}}(1-\nu_B(x)))\rangle \| x \in E\}$
$\rightarrow_{28}$	$\{\langle x, \max(\overline{\mathrm{sg}}(1-\nu_A(x)),\mu_B(x)), \min(\mu_A(x),\nu_B(x))\rangle \| x \in E\}$
$\rightarrow_{29}$	$\{\langle x, \max(\overline{\mathrm{sg}}(1-\nu_A(x)),\overline{\mathrm{sg}}(1-\mu_B(x))),$ $\min(\mu_A(x),\overline{\mathrm{sg}}(1-\nu_B(x)))\rangle \| x \in E\}$
$\rightarrow_{30}$	$\{\langle x, \max(1-\mu_A(x),\min(\mu_A(x),1-\nu_B(x))),$ $\min(\mu_A(x),\nu_B(x))\rangle \| x \in E\}$
$\rightarrow_{31}$	$\{\langle x, \overline{\mathrm{sg}}(\mu_A(x)+\nu_B(x)-1),\nu_B(x).\mathrm{sg}(\mu_A(x)+\nu_B(x)-1)\rangle \| x \in E\}$
$\rightarrow_{32}$	$\{\langle x, 1-\nu_B(x).\mathrm{sg}(\mu_A(x)+\nu_B(x)-1),$ $\nu_B(x).\mathrm{sg}(\mu_A(x)+\nu_B(x)-1)\rangle \| x \in E\}$
$\rightarrow_{33}$	$\{\langle x, 1-\min(\mu_A(x),\nu_B(x)), \min(\mu_A(x),\nu_B(x))\rangle \| x \in E\}$
$\rightarrow_{34}$	$\{\langle x, \min(1,2-\mu_A(x)-\nu_B(x)), \max(0,\mu_A(x)+\nu_B(x)-1)\rangle \| x \in E\}$
$\rightarrow_{35}$	$\{\langle x, 1-\mu_A(x).\nu_B(x), \mu_A(x).\nu_B(x)\rangle \| x \in E\}$

$\rightarrow_{36}$	$\{\langle x, \min(1-\min(\mu_A(x),\nu_B(x)),$ $\max(\mu_A(x),1-\mu_A(x)),\max(1-\nu_B(x),\nu_B(x))),$ $\max(\min(\mu_A(x),\nu_B(x)),\min(\mu_A(x),1-\mu_A(x)),$ $\min(1-\nu_B(x),\nu_B(x)))\rangle \mid x \in E\}$
$\rightarrow_{37}$	$\{\langle x, 1-\max(\mu_A(x),\nu_B(x)).\text{sg}(\mu_A(x)+\nu_B(x)-1),$ $\max(\mu_A(x),\nu_B(x)).\text{sg}(\mu_A(x)+\nu_B(x)-1)\rangle \mid x \in E\}$
$\rightarrow_{38}$	$\{\langle x, 1-\mu_A(x)+(\mu_A(x)^2.(1-\nu_B(x))),$ $\mu_A(x).(1-\mu_A(x))+\mu_A(x)^2.\nu_B(x)\rangle \mid x \in E\}$
$\rightarrow_{39}$	$\{\langle x, (1-\nu_B(x)).\overline{\text{sg}}(1-\mu_A(x))$ $+\text{sg}(1-\mu_A(x)).(\overline{\text{sg}}(\nu_B(x))+(1-\mu_A(x)).\text{sg}(\nu_B(x))),$ $\nu_B(x).\overline{\text{sg}}(1-\mu_A(x))+\mu_A(x).\text{sg}(1-\mu_A(x))$ $.\text{sg}(\nu_B(x))\rangle \mid x \in E\}$
$\rightarrow_{40}$	$\{\langle x, 1-\text{sg}(\mu_A(x)+\nu_B(x)-1), 1-\overline{\text{sg}}(\mu_A(x)+\nu_B(x)-1)\rangle \mid x \in E\}$
$\rightarrow_{41}$	$\{\langle x, \max(\overline{\text{sg}}(\mu_A(x)),1-\nu_B(x)),\min(\text{sg}(\mu_A(x)),\nu_B(x))\rangle \mid x \in E\}$
$\rightarrow_{42}$	$\{\langle x, \max(\overline{\text{sg}}(\mu_A(x)),\text{sg}(1-\nu_B(x))),$ $\min(\text{sg}(\mu_A(x)),\overline{\text{sg}}(1-\nu_B(x)))\rangle \mid x \in E\}$
$\rightarrow_{43}$	$\{\langle x, \max(\overline{\text{sg}}(\mu_A(x)),1-\nu_B(x)),\min(\text{sg}(\mu_A(x)),\nu_B(x))\rangle \mid x \in E\}$
$\rightarrow_{44}$	$\{\langle x, \max(\overline{\text{sg}}(\mu_A(x)),1-\nu_B(x)),\min(\mu_A(x),\nu_B(x))\rangle \mid x \in E\}$
$\rightarrow_{45}$	$\{\langle x, \max(\overline{\text{sg}}(\mu_A(x)),\overline{\text{sg}}(\nu_B(x))),$ $\min(\mu_A(x),\overline{\text{sg}}(1-\nu_B(x)))\rangle \mid x \in E\}$
$\rightarrow_{46}$	$\{\langle x, \max(\nu_A(x),\min(1-\nu_A(x),\mu_B(x))),$ $1-\max(\nu_A(x),\mu_B(x))\rangle \mid x \in E\}$
$\rightarrow_{47}$	$\{\langle x, \overline{\text{sg}}(1-\nu_A(x)-\mu_B(x)),$ $(1-\mu_B(x)).\text{sg}(1-\nu_A(x)-\mu_B(x))\rangle \mid x \in E\}$
$\rightarrow_{48}$	$\{\langle x, 1-(1-\mu_B(x)).\text{sg}(1-\nu_A(x)-\mu_B(x)),$ $(1-\mu_B(x)).\text{sg}(1-\nu_A(x)-\mu_B(x))\rangle \mid x \in E\}$
$\rightarrow_{49}$	$\{\langle x, \min(1,\nu_A(x)+\mu_B(x)),\max(0,1-\nu_A(x)-\mu_B(x))\rangle \mid x \in E\}$
$\rightarrow_{50}$	$\{\langle x, \nu_A(x)+\mu_B(x)-\nu_A(x).\mu_B(x),$ $1-\nu_A(x)-\mu_B(x)+\nu_A(x).\mu_B(x)\rangle \mid x \in E\}$
$\rightarrow_{51}$	$\{\langle x, \min(\max(\nu_A(x),\mu_B(x)),$ $\max(1-\nu_A(x),\nu_A(x)),\max(\mu_B(x),1-\mu_B(x))),$ $\max(1-\max(\nu_A(x),\mu_B(x)),\min(1-\nu_A(x),\nu_A(x)),$ $\min(\mu_B(x),1-\mu_B(x)))\rangle \mid x \in E\}$
$\rightarrow_{52}$	$\{\langle x, 1-(1-\min(\nu_A(x),\mu_B(x))).\text{sg}(1-\nu_A(x)-\mu_B(x)),$ $1-\min(\nu_A(x),\mu_B(x)).\text{sg}(1-\nu_A(x)-\mu_B(x))\rangle \mid x \in E\}$
$\rightarrow_{53}$	$\{\langle x, \nu_A(x)+(1-\nu_A(x))^2.\mu_B(x),$ $(1-\nu_A(x)).\nu_A(x)+(1-\nu_A(x))^2.(1-\mu_B(x))\rangle \mid x \in E\}$
$\rightarrow_{54}$	$\{\langle x, \mu_B(x).\overline{\text{sg}}(\nu_A(x))$ $+\text{sg}(\nu_A(x)).(\overline{\text{sg}}(1-\mu_B(x))+\nu_A(x).\text{sg}(1-\mu_B(x))),$ $(1-\mu_B(x)).\overline{\text{sg}}(\nu_A(x))+(1-\nu_A(x)).\text{sg}(\nu_A(x)).\text{sg}(1-\mu_B(x))\rangle \mid x \in E\}$
$\rightarrow_{55}$	$\{\langle x, 1-\text{sg}(1-\nu_A(x)-\mu_B(x)),1-\overline{\text{sg}}(1-\nu_A(x)-\mu_B(x))\rangle \mid x \in E\}$

$\rightarrow_{56}$	$\{\langle x, \max(\overline{\text{sg}}(1-\nu_A(x)), \mu_B(x)),$ $\min(\text{sg}(1-\nu_A(x)), 1-\mu_B(x))\rangle \| x \in E\}$
$\rightarrow_{57}$	$\{\langle x, \max(\overline{\text{sg}}(1-\nu_A(x)), \text{sg}(\mu_B(x))),$ $\min(\text{sg}(1-\nu_A(x)), \overline{\text{sg}}(\mu_B(x)))\rangle \| x \in E\}$
$\rightarrow_{58}$	$\{\langle x, \max(\overline{\text{sg}}(1-\nu_A(x)), \overline{\text{sg}}(1-\mu_B(x))),$ $1-\max(\nu_A(x), \mu_B(x))\rangle \| x \in E\}$
$\rightarrow_{59}$	$\{\langle x, \max(\overline{\text{sg}}(1-\nu_A(x)), \mu_B(x)),$ $(1-\max(\nu_A(x), \mu_B(x)))\rangle \| x \in E\}$
$\rightarrow_{60}$	$\{\langle x, \max(\overline{\text{sg}}(1-\nu_A(x)), \overline{\text{sg}}(1-\mu_B(x))),$ $\min(1-\nu_A(x), \overline{\text{sg}}(\mu_B(x)))\rangle \| x \in E\}$
$\rightarrow_{61}$	$\{\langle x, \max(\mu_B(x), \min(\nu_B(x), \nu_A(x))), \min(\nu_B(x), \mu_A(x))\rangle \| x \in E\}$
$\rightarrow_{62}$	$\{\langle x, \overline{\text{sg}}(\nu_B(x)-\nu_A(x)), \mu_A(x).\text{sg}(\nu_B(x)-\nu_A(x))\rangle \| x \in E\}$
$\rightarrow_{63}$	$\{\langle x, 1-(1-\nu_A(x)).\text{sg}(\nu_B(x)-\nu_A(x)),$ $\mu_A(x).\text{sg}(\nu_B(x)-\nu_A(x))\rangle \| x \in E\}$
$\rightarrow_{64}$	$\{\langle x, \mu_B(x)+\nu_B(x).\nu_A(x), \nu_B(x).\mu_A(x)\rangle \| x \in E\}$
$\rightarrow_{65}$	$\{\langle x, 1-(1-\min(\mu_B(x), \nu_A(x))).\text{sg}(\nu_B(x)-\nu_A(x)),$ $\max(\nu_B(x), \mu_A(x)).\text{sg}(\nu_B(x)-\nu_A(x)).\text{sg}(\mu_A(x)-\mu_B(x))\rangle \| x \in E\}$
$\rightarrow_{66}$	$\{\langle x, \mu_B(x)+\nu_B(x)^2.\nu_A(x),$ $\nu_B(x).\mu_B(x)+\nu_B(x)^2.\mu_A(x)\rangle \| x \in E\}$
$\rightarrow_{67}$	$\{\langle x, \nu_A(x).\overline{\text{sg}}(1-\nu_B(x))+\text{sg}(1-\nu_B(x)).(\overline{\text{sg}}(1-\nu_A(x))$ $+\mu_B(x).\text{sg}(1-\nu_A(x))),$ $\mu_A(x).\overline{\text{sg}}(1-\nu_B(x))+\nu_B(x).\text{sg}(1-\nu_B(x)).\text{sg}(1-\nu_A(x))\rangle \| x \in E\}$
$\rightarrow_{68}$	$\{\langle x, 1-(1-\nu_A(x)).\text{sg}(\nu_B(x)-\nu_A(x)),$ $\mu_A(x).\text{sg}(\nu_B(x)-\nu_A(x)).\text{sg}(\mu_A(x)-\mu_B(x))\rangle \| x \in E\}$
$\rightarrow_{69}$	$\{\langle x, 1-(1-\nu_A(x)).\text{sg}(\nu_B(x)-\nu_A(x))$ $-\mu_A(x).\overline{\text{sg}}(\nu_B(x)-\nu_A(x)).\text{sg}(\mu_A(x)-\mu_B(x)),$ $\mu_A(x).\text{sg}(\mu_A(x)-\mu_B(x))\rangle \| x \in E\}$
$\rightarrow_{70}$	$\{\langle x, \max(\overline{\text{sg}}(\nu_B(x)), \nu_A(x)), \min(\text{sg}(\nu_B(x)), \mu_A(x))\rangle \| x \in E\}$
$\rightarrow_{71}$	$\{\langle x, \max(\mu_B(x), \nu_A(x)), \min(\nu_B(x).\mu_B(x)+\nu_B(x)^2, \mu_A(x))\rangle \| x \in E\}$
$\rightarrow_{72}$	$\{\langle x, \max(\mu_B(x), \nu_A(x)), \min(1-\mu_B(x), \mu_A(x))\rangle \| x \in E\}$
$\rightarrow_{73}$	$\{\langle x, \max(1-\max(\text{sg}(\nu_B(x)), \text{sg}(1-\mu_B(x))), \nu_A(x)),$ $\min(\text{sg}(1-\mu_B(x)), \mu_A(x))\rangle \| x \in E\}$
$\rightarrow_{74}$	$\{\langle x, \max(\overline{\text{sg}}(\nu_B(x)), \text{sg}(\nu_A(x))), \min(\text{sg}(\nu_B(x)), \overline{\text{sg}}(\nu_A(x)))\rangle \| x \in E\}$
$\rightarrow_{75}$	$\{\langle x, \max(\mu_B(x), \nu_A(x).(\nu_A(x)+\mu_A(x))),$ $\min(\nu_B(x).(\nu_B(x)+\mu_B(x)), \mu_A(x).(\nu_A(x)^2+\mu_A(x))$ $+\nu_A(x).\mu_A(x))\rangle \| x \in E\}$
$\rightarrow_{76}$	$\{\langle x, \max(\mu_B(x), 1-\mu_A(x)), \min(1-\mu_B(x), \mu_A(x))\rangle \| x \in E\}$
$\rightarrow_{77}$	$\{\langle x, 1-\min(\text{sg}(1-\mu_B(x)), \overline{\text{sg}}(1-\mu_A(x))),$ $\min(\text{sg}(1-\mu_B(x)), \overline{\text{sg}}(1-\mu_A(x)))\rangle \| x \in E\}$
$\rightarrow_{78}$	$\{\langle x, \max(\overline{\text{sg}}(1-\mu_B(x)), \nu_A(x)), \min(\text{sg}(\nu_B(x)), \mu_A(x))\rangle \| x \in E\}$
$\rightarrow_{79}$	$\{\langle x, \max(\overline{\text{sg}}(1-\mu_B(x)), \text{sg}(\nu_A(x))),$ $\min(\text{sg}(\nu_B(x)), \overline{\text{sg}}(1-\mu_A(x)))\rangle \| x \in E\}$

$\to_{80}$	$\{\langle x, \max(\overline{\text{sg}}(1-\mu_B(x)), \nu_A(x)), \min(\nu_B(x), \mu_A(x))\rangle \vert x \in E\}$
$\to_{81}$	$\{\langle x, \max(\overline{\text{sg}}(1-\mu_B(x)), \overline{\text{sg}}(1-\nu_A(x))),$ $\min(\nu_B(x), \overline{\text{sg}}(1-\mu_A(x)))\rangle \vert x \in E\}$
$\to_{82}$	$\{\langle x, \max(1-\nu_B(x), \min(\nu_B(x), 1-\mu_A(x))),$ $\min(\nu_B(x), \mu_A(x))\rangle \vert x \in E\}$
$\to_{83}$	$\{\langle x, \overline{\text{sg}}(\nu_B(x)+\mu_A(x)-1), \mu_A(x).\text{sg}(\nu_B(x)+\mu_A(x)-1)\rangle \vert x \in E\}$
$\to_{84}$	$\{\langle x, 1-\mu_A(x).\text{sg}(\nu_B(x)+\mu_A(x)+1),$ $\mu_A(x).\text{sg}(\nu_B(x)+\mu_A(x)+1)\rangle \vert x \in E\}$
$\to_{85}$	$\{\langle x, 1-\nu_B(x)+\nu_B(x)^2.(1-\mu_A(x)),$ $\nu_B(x).(1-\nu_B(x))+\nu_B(x)^2\rangle \vert x \in E\}$
$\to_{86}$	$\{\langle x, (1-\mu_A(x)).\overline{\text{sg}}(1-\nu_B(x))$ $+\text{sg}(1-\nu_B(x)).\overline{\text{sg}}(\mu_A(x)+\min(1-\nu_B(x), \text{sg}(\mu_A(x)))),$ $\mu_A(x).\overline{\text{sg}}(1-\nu_B(x))+\nu_B(x).\text{sg}(1-\nu_B(x)).\text{sg}(\mu_A(x))\rangle \vert x \in E\}$
$\to_{87}$	$\{\langle x, \max(\overline{\text{sg}}(\nu_B(x)), 1-\mu_A(x)), \min(\text{sg}(\nu_B(x)), \mu_A(x))\rangle \vert x \in E\}$
$\to_{88}$	$\{\langle x, \max(\overline{\text{sg}}(\nu_B(x)), \text{sg}(1-\mu_A(x))),$ $\min(\text{sg}(\nu_B(x)), \overline{\text{sg}}(1-\mu_A(x)))\rangle \vert x \in E\}$
$\to_{89}$	$\{\langle x, \max(\overline{\text{sg}}(\nu_B(x)), 1-\mu_A(x)), \min(\nu_B(x), \mu_A(x))\rangle \vert x \in E\}$
$\to_{90}$	$\{\langle x, \max(\overline{\text{sg}}(\nu_B(x)), \overline{\text{sg}}(\mu_A(x))),$ $\min(\nu_B(x), \overline{\text{sg}}(1-\mu_A(x)))\rangle \vert x \in E\}$
$\to_{91}$	$\{\langle x, \max(\mu_B(x), \min(1-\mu_B(x), \nu_A(x))),$ $1-\max(\mu_B(x), \nu_A(x))\rangle \vert x \in E\}$
$\to_{92}$	$\{\langle x, \overline{\text{sg}}(1-\mu_B(x)-\nu_A(x)),$ $\min(1-\nu_A(x), \text{sg}(1-\mu_B(x)-\nu_A(x)))\rangle \vert x \in E\}$
$\to_{93}$	$\{\langle x, 1-\min(1-\nu_A(x), \text{sg}(1-\mu_B(x)-\nu_A(x))),$ $\min(1-\nu_A(x), \text{sg}(1-\mu_B(x)-\nu_A(x)))\rangle \vert x \in E\}$
$\to_{94}$	$\{\langle x, \mu_B(x)+(1-\mu_B(x))^2.\nu_A(x),$ $(1-\mu_B(x)).\mu_B(x)+(1-\mu_B(x))^2.(1-\nu_A(x))\rangle \vert x \in E\}$
$\to_{95}$	$\{\langle x, \min(\nu_A(x), \overline{\text{sg}}(\mu_B(x)))+\text{sg}(\mu_B(x)).(\overline{\text{sg}}(1-\nu_A(x))$ $+\min(\mu_B(x), \text{sg}(1-\nu_A(x)))),$ $\min(1-\nu_A(x), \overline{\text{sg}}(\mu_B(x)))+\min(\min(1-\mu_B(x), \text{sg}(\mu_B(x))),$ $\text{sg}(1-\nu_A(x)))\rangle \vert x \in E\}$
$\to_{96}$	$\{\langle x, \max(\overline{\text{sg}}(1-\mu_B(x)), \nu_A(x)),$ $\min(\text{sg}(1-\mu_B(x)), 1-\nu_A(x))\rangle \vert x \in E\}$
$\to_{97}$	$\{\langle x, \max(\overline{\text{sg}}(1-\mu_B(x)), \text{sg}(\nu_A(x))),$ $\min(\text{sg}(1-\mu_B(x)), \overline{\text{sg}}(\nu_A(x)))\rangle \vert x \in E\}$
$\to_{98}$	$\{\langle x, \max(\overline{\text{sg}}(1-\mu_B(x)), \nu_A(x)),$ $1-\max(\mu_B(x), \nu_A(x))\rangle \vert x \in E\}$
$\to_{99}$	$\{\langle x, \max(\overline{\text{sg}}(1-\mu_B(x)), \overline{\text{sg}}(1-\nu_A(x))),$ $\min(1-\mu_B(x), \overline{\text{sg}}(\nu_A(x)))\rangle \vert x \in E\}$
$\to_{100}$	$\{\langle x, \max(\min(\nu_A(x), \text{sg}(\mu_A(x))), \mu_B(x)),$ $\min(\min(\mu_A(x), \text{sg}(\nu_A(x))), \nu_B(x))\rangle \vert x \in E\}$

$\rightarrow_{101}$	$\{\langle x, \max(\min(\nu_A(x), \text{sg}(\mu_A(x))), \min(\mu_B(x), \text{sg}(\nu_B(x)))),$ $\min(\min(\mu_A(x), \text{sg}(\nu_A(x))), \min(\nu_B(x), \text{sg}(\mu_B(x))))\rangle \| x \in E\}$
$\rightarrow_{102}$	$\{\langle x, \max(\nu_A(x), \min(\mu_B(x), \text{sg}(\nu_B(x)))),$ $\min(\mu_A(x), \min(\nu_B(x), \text{sg}(\mu_B(x))))\rangle \| x \in E\}$
$\rightarrow_{103}$	$\{\langle x, \max(\min(1-\mu_A(x), \text{sg}(\mu_A(x))), 1-\nu_B(x)),$ $\min(\mu_A(x), \text{sg}(1-\mu_A(x)), \nu_B(x))\rangle \| x \in E\}$
$\rightarrow_{104}$	$\{\langle x, \max(\min(1-\mu_A(x), \text{sg}(\mu_A(x))),$ $\min(1-\nu_B(x), \text{sg}(\nu_B(x)))),$ $\min(\min(\mu_A(x), \text{sg}(1-\mu_A(x))),$ $\min(\nu_B(x), \text{sg}(1-\nu_B(x))))\rangle \| x \in E\}$
$\rightarrow_{105}$	$\{\langle x, \max(1-\mu_A(x), \min(1-\nu_B(x), \text{sg}(\nu_B(x)))),$ $\min(\mu_A(x), \min(\nu_B(x), \text{sg}(1-\nu_B(x))))\rangle \| x \in E\}$
$\rightarrow_{106}$	$\{\langle x, \max(\min(\nu_A(x), \text{sg}(1-\nu_A(x))), \mu_B(x)),$ $\min(\min(1-\nu_A(x), \text{sg}(\nu_A(x))), 1-\mu_B(x))\rangle \| x \in E\}$
$\rightarrow_{107}$	$\{\langle x, \max(\min(\nu_A(x), \text{sg}(1-\nu_A(x))),$ $\min(\mu_B(x), \text{sg}(1-\mu_B(x)))),$ $\min(\min(1-\nu_A(x), \text{sg}(\nu_A(x))),$ $\min(1-\mu_B(x), \text{sg}(\mu_B(x))))\rangle \| x \in E\}$
$\rightarrow_{108}$	$\{\langle x, \max(\nu_A(x), \min(\mu_B(x), \text{sg}(1-\mu_B(x)))),$ $\min(1-\nu_A(x), \min(1-\mu_B(x), \text{sg}(\mu_B(x))))\rangle \| x \in E\}$
$\rightarrow_{109}$	$\{\langle x, \nu_A(x) + \min(\overline{\text{sg}}(1-\mu_A(x)), \mu_B(x)),$ $\mu_A(x).\nu_A(x) + \min(\overline{\text{sg}}(1-\mu_A(x)), \nu_B(x))\rangle \| x \in E\}$
$\rightarrow_{110}$	$\{\langle x, \max(\nu_A(x), \mu_B(x)),$ $\min(\mu_A(x).\nu_A(x) + \overline{\text{sg}}(1-\mu_A(x)), \nu_B(x))\rangle \| x \in E\}$
$\rightarrow_{111}$	$\{\langle x, \max(\nu_A(x), \mu_B(x).\nu_B(x) + \overline{\text{sg}}(1-\mu_B(x))),$ $\min(\mu_A(x).\nu_A(x) + \overline{\text{sg}}(1-\mu_A(x)), \nu_B(x).(\mu_B(x).\nu_B(x)$ $+\overline{\text{sg}}(1-\mu_B(x))) + \overline{\text{sg}}(1-\nu_B(x)))\rangle \| x \in E\}$
$\rightarrow_{112}$	$\{\langle x, \nu_A(x) + \mu_B(x) - \nu_A(x).\mu_B(x),$ $\mu_A(x).\nu_A(x) + \overline{\text{sg}}(1-\mu_A(x)).\nu_B(x)\rangle \| x \in E\}$
$\rightarrow_{113}$	$\{\langle x, \nu_A(x) + (\mu_B(x).\nu_B(x)) - \nu_A(x).$ $(\mu_B(x).\nu_B(x) + \overline{\text{sg}}(1-\mu_B(x))),$ $(\mu_A(x).\nu_A(x) + \overline{\text{sg}}(1-\mu_A(x))).(\nu_B(x).(\mu_B(x).\nu_B(x)$ $+\overline{\text{sg}}(1-\mu_B(x))) + \overline{\text{sg}}(1-\nu_B(x)))\rangle \| x \in E\}$
$\rightarrow_{114}$	$\{\langle x, 1-\mu_A(x) + \min(\overline{\text{sg}}(1-\mu_A(x)), 1-\nu_B(x)),$ $\mu_A(x).(1-\mu_A(x)) + \min(\overline{\text{sg}}(1-\mu_A(x)), \nu_B(x))\rangle \| x \in E\}$
$\rightarrow_{115}$	$\{\langle x, 1-\min(\mu_A(x), \nu_B(x)),$ $\min(\mu_A(x).(1-\mu_A(x)) + \overline{\text{sg}}(1-\mu_A(x)), \nu_B(x))\rangle \| x \in E\}$
$\rightarrow_{116}$	$\{\langle x, \max(1-\mu_A(x), (1-\nu_B(x)).\nu_B(x) + \overline{\text{sg}}(\nu_B(x))),$ $\min(\mu_A(x).(1-\mu_A(x)) + \overline{\text{sg}}(1-\mu_A(x)),$ $\nu_B(x).((1-\nu_B(x)).\nu_B(x)$ $+\overline{\text{sg}}(\nu_B(x))) + \overline{\text{sg}}(1-\nu_B(x)))\rangle \| x \in E\}$

$\rightarrow_{117}$	$\{\langle x, 1-\mu_A(x)-\nu_B(x)+\mu_A(x).\nu_B(x)$ $(\mu_A(x).(1-\mu_A(x))+\overline{\text{sg}}(1-\mu_A(x))).\nu_B(x)\rangle \| x \in E\}$
$\rightarrow_{118}$	$\{\langle x, (1-\mu_A(x)).\text{sg}(\nu_B(x))+\mu_A(x).\nu_B(x).(1-\nu_B(x)),$ $(\mu_A(x)-\mu_A(x)^2+\overline{\text{sg}}(1-\mu_A(x))).((1-\nu_B(x)).\nu_B(x)^2$ $+\overline{\text{sg}}(1-\nu_B(x)))+\overline{\text{sg}}(1-\nu_B(x))\rangle \| x \in E\}$
$\rightarrow_{119}$	$\{\langle x, \nu_A(x)+\min(\overline{\text{sg}}(\nu_A(x)),\mu_B(x)),$ $(1-\nu_A(x)).\nu_A(x)+\min(\overline{\text{sg}}(\nu_A(x)),1-\mu_B(x))\rangle \| x \in E\}$
$\rightarrow_{120}$	$\{\langle x, \max(\nu_A(x),\mu_B(x)),$ $\min((1-\nu_A(x)).\nu_A(x)+\overline{\text{sg}}(\nu_A(x)),1-\mu_B(x))\rangle \| x \in E\}$
$\rightarrow_{121}$	$\{\langle x, \max(\nu_A(x),\mu_B(x).(1-\mu_B(x))+\overline{\text{sg}}(1-\mu_B(x))),$ $\min((1-\nu_A(x)).\nu_A(x)+\overline{\text{sg}}(\nu_A(x)),(1-\mu_B(x)).(\mu_B(x)$ $.(1-\mu_B(x))+\overline{\text{sg}}(1-\mu_B(x)))+\overline{\text{sg}}(\mu_B(x)))\rangle \| x \in E\}$
$\rightarrow_{122}$	$\{\langle x, \nu_A(x)+\mu_B(x)-\nu_A(x).\mu_B(x),$ $((1-\nu_A(x)).\nu_A(x)+\overline{\text{sg}}(\nu_A(x))).(1-\mu_B(x))\rangle \| x \in E\}$
$\rightarrow_{123}$	$\{\langle x, \nu_A(x)+\mu_B(x).(1-\mu_B(x))-\nu_A(x)$ $.(\mu_B(x).(1-\mu_B(x))+\overline{\text{sg}}(1-\mu_B(x))),$ $((1-\nu_A(x)).\nu_A(x)+\overline{\text{sg}}(\nu_A(x))).(((1-\mu_B(x)).(\mu_B(x).(1-\mu_B(x))$ $+\overline{\text{sg}}(1-\mu_B(x))))+\overline{\text{sg}}(\mu_B(x)))\rangle \| x \in E\}$
$\rightarrow_{124}$	$\{\langle x, \mu_B(x)+\min(\overline{\text{sg}}(1-\nu_B(x)),\nu_A(x)),$ $\nu_B(x).\mu_B(x)+\min(\overline{\text{sg}}(1-\nu_B(x)),\mu_A(x))\rangle \| x \in E\}$
$\rightarrow_{125}$	$\{\langle x, \max(\mu_B(x),\nu_A(x)),$ $\min(\nu_B(x).\mu_B(x)+\overline{\text{sg}}(1-\nu_B(x)),\mu_A(x))\rangle \| x \in E\}$
$\rightarrow_{126}$	$\{\langle x, \max(\mu_B(x),\nu_A(x).\mu_A(x)+\overline{\text{sg}}(1-\nu_A(x))),$ $\min(\nu_B(x).\mu_B(x)+\overline{\text{sg}}(1-\nu_B(x)),\mu_A(x).$ $(\nu_A(x).\mu_A(x)+\overline{\text{sg}}(1-\nu_A(x)))+\overline{\text{sg}}(1-\mu_A(x)))\rangle \| x \in E\}$
$\rightarrow_{127}$	$\{\langle x, \mu_B(x)+\nu_A(x)-\mu_B(x).\nu_A(x),$ $(\nu_B(x).\mu_B(x)+\overline{\text{sg}}(1-\nu_B(x))).\mu_A(x)\rangle \| x \in E\}$
$\rightarrow_{128}$	$\{\langle x, \mu_B(x)+\nu_A(x).\mu_A(x)-\mu_B(x).$ $(\nu_A(x).\mu_A(x)+\overline{\text{sg}}(1-\nu_A(x))),$ $(\nu_B(x).\mu_B(x)+\overline{\text{sg}}(1-\nu_B(x))).(\mu_A(x).(\nu_A(x).\mu_A(x)$ $+\overline{\text{sg}}(1-\nu_A(x)))+\overline{\text{sg}}(1-\mu_A(x)))\rangle \| x \in E\}$
$\rightarrow_{129}$	$\{\langle x, 1-\nu_B(x)+\min(\overline{\text{sg}}(1-\nu_B(x)),1-\mu_A(x)),$ $\nu_B(x).(1-\nu_B(x))+\min(\overline{\text{sg}}(1-\nu_B(x)),\mu_A(x))\rangle \| x \in E\}$
$\rightarrow_{130}$	$\{\langle x, 1-\min(\nu_B(x),\mu_A(x)),$ $\min(\nu_B(x).(1-\nu_B(x))+\overline{\text{sg}}(1-\nu_B(x)),\mu_A(x))\rangle \| x \in E\}$
$\rightarrow_{131}$	$\{\langle x, \max(1-\nu_B(x),(1-\mu_A(x)).\mu_A(x)+\overline{\text{sg}}(\mu_A(x))),$ $\min(\nu_B(x).(1-\nu_B(x))+\overline{\text{sg}}(1-\nu_B(x)),\mu_A(x).((1-\mu_A(x))$ $.\mu_A(x)+\overline{\text{sg}}(\mu_A(x)))+\overline{\text{sg}}(1-\mu_A(x)))\rangle \| x \in E\}$
$\rightarrow_{132}$	$\{\langle x, 1-\mu_A(x).\nu_B(x),$ $(\nu_B(x).(1-\nu_B(x))+\overline{\text{sg}}(1-\nu_B(x))).\mu_A(x)\rangle \| x \in E\}$

$\rightarrow_{133}$	$\{\langle x, 1-\nu_B(x)+(1-\mu_A(x)).\mu_A(x)$ $-(1-\nu_B(x)).((1-\mu_A(x)).\mu_A(x)+\overline{\text{sg}}(\mu_A(x))),$ $(\nu_B(x).(1-\nu_B(x))+\overline{\text{sg}}(1-\nu_B(x))).(\mu_A(x).((1-\mu_A(x)).\mu_A(x)$ $+\overline{\text{sg}}(\mu_A(x)))+\overline{\text{sg}}(1-\mu_A(x)))\rangle \mid x\in E\}$
$\rightarrow_{134}$	$\{\langle x, \mu_B(x)+\min(\overline{\text{sg}}(\mu_B(x)), \nu_A(x)),$ $(1-\mu_B(x)).\mu_B(x)+\min(\overline{\text{sg}}(\mu_B(x)), 1-\nu_A(x))\rangle \mid x\in E\}$
$\rightarrow_{135}$	$\{\langle x, \max(\mu_B(x), \nu_A(x)),$ $\min((1-\mu_B(x)).\mu_B(x)+\overline{\text{sg}}(\mu_B(x)), 1-\nu_A(x))\rangle \mid x\in E\}$
$\rightarrow_{136}$	$\{\langle x, \max(\mu_B(x), \nu_A(x).(1-\nu_A(x))+\overline{\text{sg}}(1-\nu_A(x))),$ $\min((1-\mu_B(x)).\mu_B(x)+\overline{\text{sg}}(\mu_B(x)), (1-\nu_A(x))$ $.(\nu_A(x).(1-\nu_A(x))+\overline{\text{sg}}(1-\nu_A(x)))+\overline{\text{sg}}(\nu_A(x)))\rangle \mid x\in E\}$
$\rightarrow_{137}$	$\{\langle x, \mu_B(x)+\nu_A(x)-\mu_B(x).\nu_A(x),$ $((1-\mu_B(x)).\mu_B(x)+\overline{\text{sg}}(\mu_B(x))).(1-\nu_A(x))\rangle \mid x\in E\}$
$\rightarrow_{138}$	$\{\langle x, \mu_B(x)+\nu_A(x).(1-\nu_A(x))$ $-\mu_B(x).(\nu_A(x).(1-\nu_A(x))+\overline{\text{sg}}(1-\nu_A(x))),$ $((1-\mu_B(x)).\mu_B(x)+\overline{\text{sg}}(\mu_B(x))).(1-\nu_A(x)).(\nu_A(x).(1-\nu_A(x))$ $+\overline{\text{sg}}(1-\nu_A(x))+\overline{\text{sg}}(\nu_A(x)))\rangle \mid x\in E\}$
$\rightarrow_{139}$	$\{\langle x, \max(\nu_A(x), \min(\mu_A(x), \mu_B(x))),$ $\min(\mu_A(x), \max(\nu_A(x), \nu_B(x)))\rangle \mid x\in E\}.$

Now, we discuss the following set-form of (1) for two IFSs A and B.

$$A \rightarrow B = A \rightarrow (A \rightarrow B), \tag{2}$$

Theorem 1. For every two IFSs A and B (2) is valid for implications $\rightarrow_1$, $\rightarrow_2$, $\rightarrow_3$, $\rightarrow_4$, $\rightarrow_8$, $\rightarrow_{10}$, $\rightarrow_{11}$, $\rightarrow_{12}$, $\rightarrow_{14}$, $\rightarrow_{16}$, $\rightarrow_{17}$, $\rightarrow_{18}$, $\rightarrow_{19}$, $\rightarrow_{20}$, $\rightarrow_{22}$, $\rightarrow_{23}$, $\rightarrow_{25}$, $\rightarrow_{26}$, $\rightarrow_{27}$, $\rightarrow_{28}$, $\rightarrow_{30}$, $\rightarrow_{31}$, $\rightarrow_{32}$, $\rightarrow_{33}$, $\rightarrow_{36}$, $\rightarrow_{37}$, $\rightarrow_{39}$, $\rightarrow_{40}$, $\rightarrow_{41}$, $\rightarrow_{42}$, $\rightarrow_{43}$, $\rightarrow_{48}$, $\rightarrow_{51}$, $\rightarrow_{52}$, $\rightarrow_{54}$, $\rightarrow_{55}$, $\rightarrow_{56}$, $\rightarrow_{57}$, $\rightarrow_{59}$, $\rightarrow_{61}$, $\rightarrow_{67}$, $\rightarrow_{72}$, $\rightarrow_{73}$, $\rightarrow_{74}$, $\rightarrow_{76}$, $\rightarrow_{77}$, $\rightarrow_{78}$, $\rightarrow_{79}$, $\rightarrow_{80}$, $\rightarrow_{81}$, $\rightarrow_{86}$, $\rightarrow_{87}$, $\rightarrow_{88}$, $\rightarrow_{89}$, $\rightarrow_{91}$, $\rightarrow_{92}$, $\rightarrow_{95}$, $\rightarrow_{96}$, $\rightarrow_{97}$, $\rightarrow_{100}$, $\rightarrow_{105}$, $\rightarrow_{106}$, $\rightarrow_{109}$, $\rightarrow_{110}$, $\rightarrow_{114}$, $\rightarrow_{119}$, $\rightarrow_{120}$, $\rightarrow_{139}$.

Proof. Let us check the validity of the latest case for the two fixed IFSs A and B. Let

$$Z \equiv A \rightarrow (A \rightarrow B).$$

Then,

$$Z = A \rightarrow \{\langle x, \max(\nu_A(x), \min(\mu_A(x), \mu_B(x))),$$

$$\min(\mu_A(x), \max(\nu_A(x), \nu_B(x)))\rangle \mid x \in E\}$$

$$= \{\langle x, \max(\nu_A(x), \min(\mu_A(x), \max(\nu_A(x), \min(\mu_A(x), \mu_B(x))))),$$

$$\min(\mu_A(x), \max(\nu_A(x), \min(\mu_A(x), \max(\nu_A(x), \nu_B(x)))))\rangle \mid x \in E\}.$$

1. Let $\mu_A(x) \leq \nu_A(x)$, then

$$\max(\nu_A(x), \min(\mu_A(x), ...)) = \nu_A(x)$$

and

$$\min(\mu_A(x), \max(\nu_A(x), ...)) = \mu_A(x)$$

and

$$Z = \{\langle x, \nu_A(x), \mu_A(x)\rangle | x \in E\}$$

$$= \{\langle x, \max(\nu_A(x), \min(\mu_A(x), \mu_B(x))),$$

$$\min(\mu_A(x), \max(\nu_A(x), \nu_B(x)))\rangle | x \in E\}$$

$$= A \rightarrow B.$$

2. Let $\mu_A(x) > \nu_A(x)$.
2.1. If $\mu_A(x) \leq \mu_B(x)$, then

$$Z = \{\langle x, \max(\nu_A(x), \min(\mu_A(x), \max(\nu_A(x), \mu_A(x)))),$$

$$\min(\mu_A(x), \max(\nu_A(x), \min(\mu_A(x), \max(\nu_A(x), \nu_B(x)))))\rangle | x \in E\}.$$

2.1.1. If $\nu_A(x) \leq \nu_B(x)$, then

$$Z = \{\langle x, \max(\nu_A(x), \min(\mu_A(x), \max(\nu_A(x), \mu_A(x)))),$$

$$\min(\mu_A(x), \max(\nu_A(x), \min(\mu_A(x), \nu_B(x))))\rangle | x \in E\}$$

$$= \{\langle x, \max(\nu_A(x), \mu_A(x)), \min(\mu_A(x), \min(\mu_A(x), \nu_B(x)))\rangle | x \in E\}$$

$$= \{\langle x, \max(\nu_A(x), \min(\mu_A(x), \mu_B(x))),$$

$$\min(\mu_A(x), \max(\nu_A(x), \nu_B(x)))\rangle | x \in E\}$$

$$= A \rightarrow B.$$

2.1.2. If $\nu_A(x) > \nu_B(x)$, then

$$Z = \{\langle x, \max(\nu_A(x), \mu_A(x)),$$

$$\min(\mu_A(x), \max(\nu_A(x), \min(\mu_A(x), \nu_A(x))))\rangle | x \in E\}$$

$$= \{\langle x, \max(\nu_A(x), \mu_A(x)), \min(\mu_A(x), \nu_A(x))\rangle | x \in E\}$$

$$= A \rightarrow B.$$

2.2. If $\mu_A(x) > \mu_B(x)$, then

$$Z = \{\langle x, \max(\nu_A(x), \min(\mu_A(x), \max(\nu_A(x), \min(\mu_A(x), \mu_B(x))))),$$

$$\min(\mu_A(x), \max(\nu_A(x), \min(\mu_A(x), \max(\nu_A(x), \nu_B(x)))))\rangle | x \in E\}.$$

2.2.1. If $\nu_A(x) \leq \nu_B(x)$, then

$$Z = \{\langle x, \max(\nu_A(x), \min(\mu_A(x), \max(\nu_A(x), \mu_B(x)))),$$

$$\min(\mu_A(x), \max(\nu_A(x), \min(\mu_A(x), \nu_B(x))))\rangle | x \in E\}$$

$$= \{\langle x, \max(\nu_A(x), \mu_B(x)), \min(\mu_A(x), \nu_B(x)))\rangle | x \in E\}$$

$$= A \to B.$$

2.2.1. If $\nu_A(x) > \nu_B(x)$, then

$$Z = \{\langle x, \max(\nu_A(x), \min(\mu_A(x), \max(\nu_A(x), \mu_B(x)))),$$

$$\min(\mu_A(x), \max(\nu_A(x), \min(\mu_A(x), \nu_A(x)))))\rangle | x \in E\}$$

$$= \{\langle x, \max(\nu_A(x), \mu_B(x)), \min(\mu_A(x), \nu_A(x)))\rangle | x \in E\}$$

$$= A \to B.$$

The checks of the rest equalities is similar. The proof of the next assertion – too.

Theorem 2. For every two IFSs A and B, the set

$$(A \to B) \to (A \to (A \to B)) \tag{3}$$

is an IFTS for implications $\to_1$, $\to_2$, $\to_3$, $\to_4$, $\to_5$, $\to_6$, $\to_7$, $\to_8$, $\to_9$, $\to_{11}$, $\to_{12}$, $\to_{13}$, $\to_{14}$, $\to_{17}$, $\to_{18}$, $\to_{20}$, $\to_{21}$, $\to_{22}$, $\to_{23}$, $\to_{24}$, $\to_{25}$, $\to_{27}$, $\to_{28}$, $\to_{29}$, $\to_{30}$, $\to_{31}$, $\to_{32}$, $\to_{33}$, $\to_{34}$, $\to_{35}$, $\to_{36}$, $\to_{37}$, $\to_{38}$, $\to_{40}$, $\to_{42}$, $\to_{43}$, $\to_{44}$, $\to_{45}$, $\to_{46}$, $\to_{48}$, $\to_{49}$, $\to_{50}$, $\to_{51}$, $\to_{52}$, $\to_{53}$, $\to_{55}$, $\to_{56}$, $\to_{57}$, $\to_{61}$, $\to_{62}$, $\to_{63}$, $\to_{64}$, $\to_{65}$, $\to_{66}$, $\to_{68}$, $\to_{71}$, $\to_{72}$, $\to_{74}$, $\to_{75}$, $\to_{76}$, $\to_{77}$, $\to_{79}$, $\to_{80}$, $\to_{81}$, $\to_{82}$, $\to_{83}$, $\to_{84}$, $\to_{85}$, $\to_{88}$, $\to_{89}$, $\to_{90}$, $\to_{91}$, $\to_{93}$, $\to_{94}$, $\to_{97}$, $\to_{101}$, $\to_{102}$, $\to_{103}$, $\to_{104}$, $\to_{105}$, $\to_{106}$, $\to_{107}$, $\to_{109}$, $\to_{110}$, $\to_{111}$, $\to_{112}$, $\to_{113}$, $\to_{114}$, $\to_{115}$, $\to_{116}$, $\to_{117}$, $\to_{118}$, $\to_{119}$, $\to_{120}$, $\to_{121}$, $\to_{122}$, $\to_{124}$, $\to_{125}$, $\to_{126}$, $\to_{127}$, $\to_{128}$, $\to_{129}$, $\to_{130}$, $\to_{131}$, $\to_{132}$, $\to_{133}$, $\to_{134}$, $\to_{135}$, $\to_{136}$, $\to_{137}$, $\to_{139}$.

3 Conclusion

The above theorems show the implications having standard behaviour.

In the future, it will be checked in which cases the IFS

$$(A \to (A \to B)) \to (A \to B)$$

is an IFTS. Other interesting equalities will be discussed also.

Acknowledgements

The author is grateful for the support provided by the projects DID-02-29 “Modelling processes with fixed development rules” and BIn-02/09 “Design and development of intuitionistic fuzzy logic tools in information technologies” funded by the National Science Fund, Bulgarian Ministry of Education, Youth and Science.

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