On two concepts of probability on IF-sets

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Abstract. In the paper the concept based on the Lukasiewicz connectives is compared with new one using the max and min operations.

1 Introduction

Probably the first definition of the probability on a family $\mathcal{F}$ of IF events was given by Grzegorzewski and Mrovka ([1])as a function from $\mathcal{F}$ to the family $\mathcal{J}$ of compact intervals. Using the Lukasiewicz connectives an axiomatic definition of the probability was given in [3]. Moreover, on [4] a general form of such probabilities was presented. Recently a new axiomatic approach based on max - min operations appeared ([2]). Here we first explain the main results of our theory (Section 2) and then we discuss the intuitionistic range (Section 3) and the intuitionistic domain (Section 4).

2 Lukasiewicz connectives

Consider a classical probability space $(\Omega, \mathcal{S}, P)$ and the family $\mathcal{F}$ of all IF events $A = (\mu_A, \nu_A)$, i.e. $\mu_A, \nu_A : \Omega \to [0,1]$ are $\mathcal{S}$-measurable, and $\mu_A + \nu_A \leq 1$. In [1] the following mapping $\mathcal{P} : \mathcal{F} \to \mathcal{J}$ was constructed

$$\mathcal{P}(A) = [\int \mu_A dP, 1 - \int \nu_A dP].$$

Denote by $\mathcal{J}$ the family of all compact intervals on the real line. Our axioms are based on the Lukasiewicz connectives

$$A \oplus B = (\mu_A \oplus \mu_B, \nu_A \odot \nu_B)$$

$$A \odot B = (\mu_A \odot \mu_B, \nu_A \oplus \nu_B),$$

where

$$f \oplus g = \min(f+g, 1), f \odot g = \max(f+g-1, 0).$$

Our axioms for the probability $\mathcal{P} : \mathcal{F} \to \mathcal{F}$ are the following:

(i) $\mathcal{P}((1_\Omega, 0_\Omega)) = [1,1], \mathcal{P}((0_\Omega, 1_\Omega)) = [0,0]$;

(ii) $A \odot B = (0_\Omega, 1_\Omega) \Longrightarrow \mathcal{P}(A \oplus B) = \mathcal{P}(A) + \mathcal{P}(B)$;

(iii) If $A_n \nearrow A(i.e. \mu_{A_n} \nearrow \mu_A, \nu_{A_n} \searrow \nu_A)$, then

$$\mathcal{P}(A_n) = [\mathcal{P}^\flat(A_n), \mathcal{P}^\sharp(A_n)] \nearrow \mathcal{P}(A) = [\mathcal{P}^\flat(A), \mathcal{P}^\sharp(A)],$$

i.e. $\mathcal{P}^\flat(A_n) \nearrow \mathcal{P}^\flat(A), \mathcal{P}^\sharp(A_n) \nearrow \mathcal{P}^\sharp(A)$.

The representation theorem says that to any $\mathcal{P}$ there are $\alpha, \beta \in [0,1], \alpha \leq \beta$ and a probability $m : \mathcal{S} \to [0,1]$ such that

$$\mathcal{P}(A) = [(1-\alpha)\int \mu_A dm + \alpha(1 - \int \nu_A dm), (1-\beta)\int \mu_A dm + \beta(1 - \int \nu_A dm)].$$

3 Intuitionistic range

In [2] M. Krachounov starts with a σ-algebra $\mathcal{S}$ of subsets of Ω and a mapping $P_I : \mathcal{S} \to [0,1] \times [0,1]$

$$P_I(A) = [P(A), Q(A)], A \in \mathcal{S}.$$

He gives axioms separately for the left side $P : \mathcal{S} \to [0,1]$ and the right side $Q : \mathcal{S} \to [0,1]$:

1. $P(A) \geq 0, Q(A) \leq 1$, and $P(A) + Q(A) \leq 1$ for every $A \in \mathcal{S}$.
2. $P(\Omega) = p \leq 1$, and $Q(\Omega) = 0$.
3. When $A \cap B = \emptyset$, then $P(A \cup B) = P(A) + P(B)$ and $Q(A \cup B) = Q(A) + Q(B) - q$, where $q = Q(\emptyset) \in (0,1]$.
4. If (A_n) is a sequence of events and $A_n \searrow \emptyset$, then

$$\lim_{n\to\infty} P(A_n) = 0, \lim_{n\to\infty} Q(A_n) = q.$$

For simplicity put $P(\Omega) = 1, Q(\emptyset) = 1$ and compare it with the Grzegorzewski and Mrowka definition

$$\mathcal{P}(A) = [\int \mu_A dm, 1 - \int \nu_A dm].$$

Recall that we have in the Krachounov definition a special case. To any $A \in \mathcal{S}$ one can consider the pair $(\chi_A, 1 - \chi_A) = (\chi_A, \chi_{A'})$. Starting with the formula $\mathcal{P}(A) = [\mathcal{P}^\flat(A), \mathcal{P}^\sharp(A)]$ we consider two functions

$$\mathcal{P}^\flat : \mathcal{S} \to [0,1], \mathcal{P}^\sharp : \mathcal{S} \to [0,1].$$

Here $\mathcal{P}^\flat(A)$ corresponds to $\int \chi_A dm = m(A)$, $\mathcal{P}^\sharp(A)$ corresponds to $1 - \int \nu_A dm = 1 - \int (1-\chi_A) dm = m(A)$. On the other hand the Krachounov $P : \mathcal{S} \to [0,1]$ corresponds to the same part $\int \mu_A dm$, hence $\mathcal{P}^\flat$ and P satisfy the same axioms, but in the right side he consider $\int \nu_A dm$, not $1 - \int \nu_A dm$, hence

$$\int \nu_A dm = \int (1 - \chi_A) dm = m(A').$$

Therefore

$$Q(A) = 1 - \mathcal{P}^\sharp(A),$$

and our additivity is in duce with his

$$A \cap B = \emptyset \Longrightarrow Q(A \cup B) = 1 - \mathcal{P}^\sharp(A \cup B)$$
$$= 1 - \mathcal{P}^\sharp(A) - \mathcal{P}^\sharp(B)$$
$$= 1 - \mathcal{P}^\sharp(A) + 1 - \mathcal{P}^\sharp(B) - 1$$
$$= Q(A) + Q(B) - 1.$$

The reason of our approach was in the definition of $\mathcal{P}$ as a mapping with the range $\mathcal{J}$ and not as a couple of two mappings P and Q with the range [0,1]. Of course, these two approaches are equivalent.

4 Intuitionistic domain

Since every compact interval $[a, b]$ is determined by its endpoints a, b, instead of defining probability from $\mathcal{F}$ to $\mathcal{J}$ it is reasonable to consider only the functions $P : \mathcal{F} \to [0, 1]$. Of course, there are two concepts of additivity: Krachounov's and ours. The Krachounov definition definition of additivity is the following:

$$(*) P(A) + P(B) = P(A \cup B) + P(A \cap B).$$

Our definition is the following:

$$(**) P(A) + P(B) = P(A \oplus B) + P(\odot B).$$

Formally (**) is stronger than(ii), of course, as a simple consequence of the representation theorem one can find that (**) and (ii) are equivalent for continuous probabilities. Moreover, we shall show that (*) follows from (**).

Theorem 4.1 *If $\mathcal{P} : \mathcal{F} \to [0, 1]$ is a continuous probability satisfying (**), then $\mathcal{P}$ satisfies also (*).*

Proof. By the representation theorem ([4]) there exists $\alpha \in [0, 1]$ and a probability measure $m : \mathcal{S} \to [0, 1]$ such that

$$\mathcal{P}(A) = (1 - \alpha) \int \mu_A dm + \alpha(1 - \int \nu_A dm).$$

Recall that

$$\mu_{A \cup B} = \mu_A \vee \mu_B = \mu_A + \mu_B - \mu_A \wedge \mu_B = \mu_A + \mu_B - \mu_{A \cap B}$$

$$\nu_{A \cup B} = \nu_A \wedge \nu_B = \nu_A + \nu_B - \nu_A \vee \nu_B = \nu_A + \nu_B - \nu_{A \cap B}.$$

Moreover

$$P(B) = (1 - \alpha) \int \mu_B dm + \alpha(1 - \int \nu_B dm).$$

Therefore

$$P(A)+P(B)=(1-\alpha)\int(\mu_A+\mu_B)dm+(1-\int\nu_A dm+1-\int\nu_B dm)$$

$$=(1-\alpha)\int(\mu_{A\cup B}+\mu_{A\cap B})dm+\alpha(1-\int\nu_{A\cup B}dm+1-\int\nu_{A\cap B}dm)$$

$$=(1-\alpha)\int\mu_{A\cup B}+\alpha(1-\int\nu_{A\cup B}dm)+(1-\alpha)\int\mu_{A\cap B}dm+\alpha(1-\int\nu_{A\cap B}dm)=$$

$$=P(A\cup B)+P(A\cap B).$$

Conclusion. We proved that the Krachounov theory is more general. Of course, the theory needs also the notion of observable and some theorems of the type of large numbers and central limit theorem. It would be a subject of some further research.

Ackowledgement. The paper was supported by Grant VEGA 1/2002/05.

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