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Continuity of intuitionistic fuzzy proper functions on intuitionistic smooth fuzzy topological spaces

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Abstract

The intuitionistic fuzzy proper function is introduced and the relations among various types of continuity of intuitionistic fuzzy proper function on an intuitionistic fuzzy set and at every intuitionistic fuzzy point belonging to the intuitionistic fuzzy set in the context of intuitionistic smooth fuzzy topological spaces are discussed. The projection maps are also defined as intuitionistic fuzzy proper functions and their properties are proved.

Keywords: Intuitionistic fuzzy proper function, intuitionistic smooth fuzzy topology, intuitionistic smooth fuzzy continuity.

Mathematics Subject Classification: 03E72, 54A40.

1 Introduction

Chang introduced topology of fuzzy subsets in [4] and Šostak introduced the fuzzy topology in [19, 20]. Šostak's topology is latter redefined by various researchers and is called *smooth fuzzy topology* [18]. In the context of smooth fuzzy topological spaces, neighborhood structures [9], base and subbase [16], product topology [21], compactness [2, 10, 11, 12], separation axioms [21], gradation preserving functions [14] are also studied. On the other hand, Atanassov [1] generalized the fuzzy subsets to intuitionistic fuzzy subsets. Later Çoker [6] defined intuitionistic fuzzy topological spaces, fuzzy continuity and some other concepts in intuitionistic fuzzy topological spaces. In [15], Mondal and Samanta introduced gradation of openness on collection of fuzzy subsets. Çoker and Demirci [8] introduced the basic definitions and properties of intuitionistic fuzzy topological spaces in Šostak sense.

Regarding functions in fuzzy setting, the fuzzy proper function and its continuity on Chang topological spaces are first introduced by Chakraborty and Ahsanulla [3]. Chaudhuri and Das [5] proved the equivalent statements of continuity of fuzzy proper function in the context of Chang topology. Fath Allah and Mahmoud [13] introduced the fuzzy graph, strongly fuzzy graph of a fuzzy proper function on Chang topological space. They proved the closed graph theorem under some sufficient conditions and also proved various results relating separation axioms, the continuity of fuzzy proper function and the closedness of its graph. The notions of smooth fuzzy continuity and weakly smooth fuzzy continuity of a fuzzy proper function on smooth fuzzy topological spaces and their properties are discussed in [18].

This paper is organized as follows. In Section 2, we recall some basic definitions and results from the literature and also prove some preliminary results required for our discussion.

The section 3 is devoted to discuss intuitionistic smooth fuzzy continuity and intuitionistic weak smooth fuzzy continuity in intuitionistic smooth fuzzy topological spaces. As in the classical case of continuity of functions between two topological spaces, we prove that fuzzy continuity on a fuzzy set is equivalent to the fuzzy continuity at every fuzzy point of the fuzzy set. When we study the intuitionistic weak continuity of a intuitionistic fuzzy proper function at a intuitionistic fuzzy point, obviously, we have to consider two notions, one is defined in terms of fuzzy neighborhoods and the other in terms of quasi neighborhoods. We establish that intuitionistic weak fuzzy continuity on an intuitionistic fuzzy set implies intuitionistic weak fuzzy continuity at every intuitionistic fuzzy point in the intuitionistic fuzzy set, in both notions, and neither of the converse is true, which is in contrast with the classical case. To get the converse of one of these results, we introduce two new notions, namely, intuitionisttic α-weakly smooth fuzzy continuous functions and positive minimum intuitionistic smooth fuzzy topological spaces. We also proved that intuitionistic $(0,1)$-weakly fuzzy continuity on an intuitionistic fuzzy set implies intuitionistic $(0,1)$-weakly fuzzy continuity at every intuitionistic fuzzy point in the intuitionistic fuzzy set and the converse is not true. To get the converse, we introduce intuitionisttic (α,β)-weakly smooth fuzzy continuous functions.

In section 4, the projection functions from a product of intuitionistic fuzzy sets into a intuitionistic fuzzy set are defined as intuitionistic fuzzy proper functions, and it is proved that they are intuitionistic smooth fuzzy continuous, intuitionistic weakly smooth fuzzy continuous, intuitionistic qn-weakly smooth fuzzy continuous, intuitionistic $(0,1)$-weakly smooth fuzzy continuous, intuitionistic α-weakly smooth fuzzy continuous and intuitionistic (α,β)-weakly smooth fuzzy continuous. We also prove that every intuitionistic fuzzy proper function $\mathbb{F} : \mathbb{A} \to \prod \mathbb{B}_j$ can be expressed as $[\mathbb{F}_j]$ for some suitable $\mathbb{F}_j : \mathbb{A} \to \mathbb{B}_j$ and study the relationship between continuity of $\mathbb{F}$ and that of its coordinate functions $\mathbb{F}_j$.

2 Preliminaries

Intuitionistic fuzzy set is introduced by Atanassov in [1].

Definition 2.1 *Let X be a nonempty set and I be the closed unit interval $[0,1]$. An intuitionistic fuzzy set (IFS) $\mathbb{A}$ is a function $\mathbb{A} = (\mu_{\mathbb{A}}, \nu_{\mathbb{A}}) : X \to \zeta$ where $\zeta = \{(r,s) \in I \times I : r + s \leq 1$.*

The set of all intuitionistic fuzzy subsets of X is denoted by ζ^X.

Definition 2.2 *Let (r,s), $(p,q) \in \zeta$. We define*

1. *$(r,s) \sqcup (p,q) = (r \vee p, s \wedge q)$, where $r \vee p$ and $s \wedge q$ usual maximum of r and p, usual minimum of s and q in the ordered set of real numbers.*

2. $(r, s) \sqcap (p, q) = (r \wedge p, s \vee q)$, *where* $r \wedge p$ *and* $s \vee q$ *usual minimum of* r *and* p, *usual maximum of* s *and* q *in the ordered set of real numbers.*

3. $(r, s) \ll (p, q)$ *if* $r \leq p$ *and* $s \geq q$

4. $(r, s) \gg (p, q)$ *if* $r \geq p$ *and* $s \leq q$.

It is easy to observe that $\sqcap$ and $\sqcup$ satisfy all the laws satisfied by $\wedge$ and $\vee$.

Definition 2.3 *[1] Let* $\mathbb{A}, \mathbb{B} \in \zeta^X$. *We say that* $\mathbb{A}$ *is contained in* $\mathbb{B}$ *if* $\mathbb{A}(x) \ll \mathbb{B}(x), \forall x \in X$. *In this case, we simply write* $\mathbb{A} \ll \mathbb{B}$.

For $\mathbb{A} \in \zeta^X$, we denote $\{\mathbb{U} \in \zeta^X : \mathbb{U} \ll \mathbb{A}\}$ by $\mathscr{I}_{\mathbb{A}}$.

Definition 2.4 *[7] Let* X *be a nonempty set and* $x \in X$. *If* $r \in (0, 1]$, $s \in [0, 1)$ *such that* $r + s \leq 1$, *then the IFS*

$$\mathbb{P}_x^{(r,s)}(y) = \begin{cases} (r, s), & \text{if } y = x \\ (0, 1), & \text{otherwise} \end{cases}$$

is called an intuitionistic fuzzy point (IFP) in X.

The IFP $\mathbb{P}_x^{(r,s)}$ is said to belong to the IFS $\mathbb{A}$ if $\mathbb{P}_x^{(r,s)} \ll \mathbb{A}$. We also write this by $\mathbb{P}_x^{(r,s)} \in \mathbb{A}$.

Definition 2.5 *[7] (i) An IFP* $\mathbb{P}_x^{(r,s)}$ *in* X *is said to be quasi-coincident with an IFS* $\mathbb{A}$, *denoted by* $\mathbb{P}_x^{(r,s)} q \mathbb{A}$, *if* $r > \nu_{\mathbb{A}}(x)$ *or* $s < \mu_{\mathbb{A}}(x)$.
(ii) The IFS $\mathbb{A}$ *is said to be quasi-coincident with another IFS* $\mathbb{B}$, *denoted by* $\mathbb{A} q \mathbb{B}$ *if there exists an element* $x \in X$ *such that* $\mu_{\mathbb{A}}(x) > \nu_{\mathbb{B}}(x)$ *or* $\nu_{\mathbb{A}}(x) < \mu_{\mathbb{B}}(x)$.

Now we introduce the quasi coincidence of $\mathbb{U}, \mathbb{V} \in \mathscr{I}_{\mathbb{A}}$ related to $\mathbb{A}$ as follows.

Definition 2.6 *Let* $\mathbb{A}$ *be an IFS and* $\mathbb{U}, \mathbb{V} \in \mathscr{I}_{\mathbb{A}}$. $\mathbb{U}$ *and* $\mathbb{V}$ *are said to be quasi-coincident related to* $\mathbb{A}$, *denoted by* $\mathbb{U} q \mathbb{V}[\mathbb{A}]$ *if* $\mu_{\mathbb{U}}(x) \vee \nu_{\mathbb{A}}(x) > \mu_{\mathbb{A}}(x) \wedge \nu_{\mathbb{V}}(x)$ *or* $\nu_{\mathbb{U}}(x) \wedge \mu_{\mathbb{A}}(x) < \nu_{\mathbb{A}}(x) \vee \mu_{\mathbb{V}}(x)$, *for some* $x \in X$.

Note that $\mathbb{U} q \mathbb{V}[\mathbb{A}]$ if and only if $\mathbb{V} q \mathbb{U}[\mathbb{A}]$, and it is consistent with Definition 2.5(ii).

Definition 2.7 *Let* $\mathbb{A}$ *be an IFS and* $\mathbb{U} \in \mathscr{I}_{\mathbb{A}}$. *An IFP* $\mathbb{P}_x^{(r,s)} \in \mathbb{A}$ *is said to be quasi-coincident with* $\mathbb{U}$ *in* $\mathbb{A}$, *denoted by* $\mathbb{P}_x^{(r,s)} q \mathbb{U}[\mathbb{A}]$, *if* $\mu_{\mathbb{U}}(x) \vee \nu_{\mathbb{A}}(x) > \mu_{\mathbb{A}}(x) \wedge s$ *or* $\nu_{\mathbb{U}}(x) \wedge \mu_{\mathbb{A}}(x) < \nu_{\mathbb{A}}(x) \vee r$.

Definition 2.8 *[2] Let* $\mathbb{A} \in \zeta^X$ *and* $\mathbb{B} \in \zeta^Y$. *Define* $\mathbb{A} \times \mathbb{B} \in \zeta^{X \times Y}$ *by*

$$(\mathbb{A} \times \mathbb{B})(x, y) = \mathbb{A}(x) \sqcap \mathbb{B}(y), \forall (x, y) \in X \times Y.$$

Now we introduce intuitionistic fuzzy proper function between intuitionistic fuzzy sets as follows.

Definition 2.9 (Cf. [3]) *A intuitionistic fuzzy subset* $\mathbb{F}$ *of* $X \times Y$ *is said to be a intuitionistic fuzzy proper function from the intuitionistic fuzzy set* $\mathbb{A}$ *to intuitionistic fuzzy set* $\mathbb{B}$ *if*

1. $\mathbb{F}(x, y) \ll (\mathbb{A} \times \mathbb{B})(x, y)$ *for each* $(x, y) \in X \times Y$,

2. *for each* $x \in X$*, there exists a unique* $y_0 \in Y$ *such that* $\mathbb{F}(x, y_0) = \mathbb{A}(x)$ *and* $\mathbb{F}(x, y) = (0, 1)$ *if* $y \neq y_0$*.*

Remark 2.10 *Hereafter* $\mathbb{F} : \mathbb{A} \to \mathbb{B}$ *means that* $\mathbb{F}$ *is an intuitionistic fuzzy proper function from* $\mathbb{A} \in \zeta^X$ *into* $\mathbb{B} \in \zeta^Y$*.*

Definition 2.11 *Let* $\mathbb{F} : \mathbb{A} \to \mathbb{B}$*. If* $\mathbb{U} \ll \mathbb{A}$ *and* $\mathbb{V} \ll \mathbb{B}$*, then* $\mathbb{F}^{-1}(\mathbb{V}) \ll \mathbb{A}$*, and* $\mathbb{F}(\mathbb{U}) \ll \mathbb{B}$ *are defined by*

$$\mathbb{F}^{-1}(\mathbb{V})(x) = \bigsqcup_{s \in Y} \{\mathbb{F}(x, s) \sqcap \mathbb{V}(s)\}, \forall x \in X.$$

$$\mathbb{F}(\mathbb{U})(y) = \bigsqcup_{t \in X} \{\mathbb{F}(t, y) \sqcap \mathbb{U}(t)\}, \forall y \in Y.$$

The following lemma shows that the image of an IFP is also an IFP.

Lemma 2.12 *If* $\mathbb{P}_x^{(r,s)} \in \mathbb{A}$ *and* $\mathbb{F} : \mathbb{A} \to \mathbb{B}$*, then* $\mathbb{F}(\mathbb{P}_x^{(r,s)}) = \mathbb{P}_y^{(r,s)}$*, where* $y \in Y$ *is unique such that* $\mathbb{F}(x, y) = \mathbb{A}(x)$*.*

Proof. For $w \in Y$,

$$\begin{aligned}
\mathbb{F}(\mathbb{P}_x^{(r,s)})(w) &= \bigsqcup_{t \in X} \left\{\mathbb{F}(t, w) \sqcap \mathbb{P}_x^{(r,s)}(t)\right\} \\
&= F(x, w) \sqcap (r, s) \\
&= \begin{cases} \mathbb{A}(x) \sqcap (r, s) \text{ if } w = y \\ (0, 1) \text{ otherwise} \end{cases} \\
&= \begin{cases} (\mu_{\mathbb{A}}(x), \nu_{\mathbb{A}}(x)) \sqcap (r, s) \text{ if } w = y \\ (0, 1) \text{ otherwise} \end{cases} \\
&= \begin{cases} (r, s) \text{ if } w = y \\ (0, 1) \text{ otherwise} \end{cases} \qquad [\text{ Since } \mathbb{P}_x^{(r,s)} \in \mathbb{A}] \\
&= \mathbb{P}_y^{(r,s)}(w).
\end{aligned}$$

Hence image of an IFP is also an IFP. □

Next we present an easy formula for finding inverse image of an intuitionistic fuzzy set under an intuitionistic fuzzy proper function.

Lemma 2.13 *Let* $\mathbb{F} : \mathbb{A} \to \mathbb{B}$ *be an intuitionistic fuzzy proper function. If* $\mathbb{V} \ll \mathbb{B}$ *then* $\mathbb{F}^{-1}(\mathbb{V})(x) = \mathbb{A}(x) \sqcap \mathbb{V}(y)$*, where* $y \in Y$ *is unique such that* $\mathbb{F}(x, y) = \mathbb{A}(x)$*.*

Lemma 2.14 *Let* $\mathbb{F} : \mathbb{A} \to \mathbb{B}$*,* $\mathbb{V} \in \mathscr{I}_{\mathbb{B}}$ *and* $\mathbb{U} \in \mathscr{I}_{\mathbb{A}}$*. Then* $\mathbb{F}(\mathbb{F}^{-1}(\mathbb{V})) \ll \mathbb{V}$ *and* $\mathbb{F}^{-1}(\mathbb{F}(\mathbb{U})) \gg \mathbb{U}$*.*

Proof. Let $y \in Y$ be arbitrary.

$$\begin{aligned}
\mathbb{F}(\mathbb{F}^{-1}(\mathbb{V}))(y) &= \bigsqcup_{t \in X} \left\{\mathbb{F}(t, y) \sqcap \mathbb{F}^{-1}(\mathbb{V})(t)\right\} \\
&= \bigsqcup_{t \in X} \left\{\mathbb{F}(t, y) \sqcap \bigsqcup_{s \in Y} \{\mathbb{F}(t, s) \sqcap \mathbb{V}(s)\}\right\} \\
&\ll \bigsqcup_{t \in X} \{\mathbb{F}(t, y) \sqcap \mathbb{F}(t, y) \sqcap \mathbb{V}(y)\} \ll \mathbb{V}(y).
\end{aligned}$$

Let $x \in X$ be arbitrary and y be unique such that $\mathbb{F}(x,y) = \mathbb{A}(x)$.

$$\begin{aligned}
\mathbb{F}^{-1}(\mathbb{F}(\mathbb{U}))(x) &= \bigsqcup_{s\in Y} \{\mathbb{F}(x,s) \sqcap \mathbb{F}(\mathbb{U})(s)\} \\
&= \mathbb{F}(x,y) \sqcap \mathbb{F}(\mathbb{U})(y) \\
&= \mathbb{A}(x) \sqcap \bigsqcup_{t\in X} \{\mathbb{F}(t,y) \sqcap \mathbb{U}(t)\} \\
&\gg \mathbb{A}(x) \sqcap \mathbb{A}(x) \sqcap \mathbb{U}(x) \\
&= \mathbb{U}(x).
\end{aligned}$$

□

Lemma 2.15 *Let $\mathbb{F} : \mathbb{A} \to \mathbb{B}$ and $\mathbb{P}_x^{(r,s)} \in \mathbb{A}$. If $\mathbb{V} \ll \mathbb{B}$ and $\mathbb{F}(\mathbb{P}_x^{(r,s)}) \in \mathbb{V}$, then $\mathbb{P}_x^{(r,s)} \in \mathbb{F}^{-1}(\mathbb{V})$.*

Proof. $\mathbb{F}(\mathbb{P}_x^{(r,s)}) = \mathbb{P}_y^{(r,s)} \in \mathbb{V}$ implies that $(r,s) \ll \mathbb{V}(y)$.

$$\begin{aligned}
\mathbb{F}^{-1}(\mathbb{V})(x) &= \mathbb{A}(x) \sqcap \mathbb{V}(y), \text{ where } y \in Y \text{ is such that } \mathbb{F}(x,y) = \mathbb{A}(x). \\
&\gg \mathbb{A}(x) \sqcap (r,s) = (r,s).[\text{ since } \mathbb{A}(x) \gg (r,s)]
\end{aligned}$$

Thus $\mathbb{P}_x^{(r,s)} \in \mathbb{F}^{-1}(\mathbb{V})$. □

Lemma 2.16 *Let $\mathbb{F} : \mathbb{A} \to \mathbb{B}$ and $\mathbb{P}_x^{(r,s)} \in \mathbb{A}$. If $\mathbb{V} \ll \mathbb{B}$ and $\mathbb{F}(\mathbb{P}_x^{(r,s)}) q \mathbb{V}[\mathbb{B}]$, then $\mathbb{P}_x^{(r,s)} q \mathbb{F}^{-1}(\mathbb{V})[\mathbb{A}]$.*

Proof. We know that $\mathbb{F}(\mathbb{P}_x^{(r,s)}) = \mathbb{P}_y^{(r,s)}$, where y is unique such that $\mathbb{F}(x,y) = \mathbb{A}(x)$. From $\mathbb{F}(\mathbb{P}_x^{(r,s)}) q \mathbb{V}[\mathbb{B}]$, we have

$$\mu_{\mathbb{V}}(y) \vee \nu_{\mathbb{B}}(y) > \mu_{\mathbb{B}}(y) \wedge s \text{ or } \nu_{\mathbb{V}}(y) \wedge \mu_{\mathbb{B}}(y) < \nu_{\mathbb{B}}(y) \vee r.$$

We know that $\mathbb{F}^{-1}(\mathbb{V})(x) = (\mu_{\mathbb{A}}(x) \wedge \mu_{\mathbb{V}}(y), \nu_{\mathbb{A}}(x) \wedge \nu_{\mathbb{V}}(y))$, where $y \in Y$ is unique such that $\mathbb{F}(x,y) = \mathbb{A}(x)$. We shall show that one of the following inequalities is true:

$$(\mu_{\mathbb{A}}(x) \wedge \mu_{\mathbb{V}}(y)) \vee \nu_{\mathbb{A}}(x) > \mu_{\mathbb{A}}(x) \wedge s \text{ or } (\nu_{\mathbb{A}}(x) \vee \nu_{\mathbb{V}}(y)) \wedge \mu_{\mathbb{A}}(x) < \nu_{\mathbb{A}}(x) \vee r.$$

Case 1: $\mu_{\mathbb{V}}(y) \vee \nu_{\mathbb{B}}(y) > \mu_{\mathbb{B}}(y) \wedge s$.

$$\begin{aligned}
(\mu_{\mathbb{A}}(x) \wedge \mu_{\mathbb{V}}(y)) \vee \nu_{\mathbb{A}}(x) &= (\mu_{\mathbb{A}}(x) \vee \nu_{\mathbb{A}}(x)) \wedge (\mu_{\mathbb{V}}(y) \vee \nu_{\mathbb{A}}(x)) \\
&> (\mu_{\mathbb{A}}(x) \vee \nu_{\mathbb{A}}(x)) \wedge (\mu_{\mathbb{V}}(y) \vee \nu_{\mathbb{B}}(y)) \\
&> \mu_{\mathbb{A}}(x) \wedge \mu_{\mathbb{B}}(y) \wedge s = \mu_{\mathbb{A}}(x) \wedge s.
\end{aligned}$$

Case 2: $\nu_{\mathbb{V}}(y) \wedge \mu_{\mathbb{B}}(y) < \nu_{\mathbb{B}}(y) \vee r$.

$$\begin{aligned}
(\nu_{\mathbb{A}}(x) \vee \nu_{\mathbb{V}}(y)) \wedge \mu_{\mathbb{A}}(x) &= (\nu_{\mathbb{A}}(x) \wedge \mu_{\mathbb{A}}(x)) \vee (\nu_{\mathbb{V}}(y) \wedge \mu_{\mathbb{A}}(x)) \\
&< (\nu_{\mathbb{A}}(x) \wedge \mu_{\mathbb{A}}(x)) \vee (\nu_{\mathbb{V}}(y) \wedge \mu_{\mathbb{B}}(y)) \\
&< \nu_{\mathbb{A}}(x) \vee \nu_{\mathbb{B}}(y) \vee r = \nu_{\mathbb{A}}(x) \vee r.
\end{aligned}$$

Therefore $\mathbb{P}_x^{(r,s)} q \mathbb{F}^{-1}(\mathbb{V})[\mathbb{A}]$. □

Adopting the definition of composition of fuzzy proper functions given in [3], we define the composition of intuitionistic fuzzy proper functions as follows.

Definition 2.17 *Let $\mathbb{F}:\mathbb{A}\to\mathbb{B}$ and $\mathbb{G}:\mathbb{B}\to\mathbb{C}$ be intuitionistic fuzzy proper functions, where $C\in\zeta^Z$. Then $\mathbb{G}\circ\mathbb{F}$ is an intuitionistic fuzzy proper function from $\mathbb{A}$ to $\mathbb{C}$ defined by*

$$(\mathbb{G}\circ\mathbb{F})(x,z)=\bigsqcup_{y\in Y}\{\mathbb{F}(x,y)\sqcap\mathbb{G}(y,z)\},\forall(x,z)\in X\times Z.$$

Theorem 2.18 *Let $\mathbb{F}:\mathbb{A}\to\mathbb{B}$ and $\mathbb{G}:\mathbb{B}\to\mathbb{C}$ be intuitionistic fuzzy proper functions. Then $(\mathbb{G}\circ\mathbb{F})^{-1}(\mathbb{U})=\mathbb{F}^{-1}(\mathbb{G}^{-1}(\mathbb{U}))$, $\forall\mathbb{U}\ll\mathbb{C}$.*

Proof. Let $\mathbb{A}\in\zeta^X$, $\mathbb{B}\in\zeta^Y$ and $\mathbb{C}\in\zeta^Z$. Let $\mathbb{W}\ll\mathbb{C}$ and $x\in X$.

$$\begin{aligned}(\mathbb{G}\circ\mathbb{F})^{-1}(\mathbb{W})(z) &= \bigsqcup_{z\in Z}\{(\mathbb{G}\circ\mathbb{F})(x,z)\sqcap\mathbb{W}(z)\}\\ &= \bigsqcup_{z\in Z}\left\{\bigsqcup_{y\in Y}\{\mathbb{F}(x,y)\sqcap\mathbb{G}(y,z)\}\sqcap\mathbb{W}(z)\right\}\\ &= \bigsqcup_{z\in Z}\left\{\bigsqcup_{y\in Y}\{\mathbb{F}(x,y)\sqcap\mathbb{G}(y,z)\sqcap\mathbb{W}(z)\}\right\}\\ &= \bigsqcup_{y\in Y}\left\{\bigsqcup_{z\in Z}\{\mathbb{F}(x,y)\sqcap\mathbb{G}(y,z)\sqcap\mathbb{W}(z)\}\right\}\\ &= \bigsqcup_{y\in Y}\left\{\mathbb{F}(x,y)\sqcap\bigsqcup_{z\in Z}\{\mathbb{G}(y,z)\sqcap\mathbb{W}(z)\}\right\}\\ &= \bigsqcup_{y\in Y}\{\mathbb{F}(x,y)\sqcap\mathbb{G}^{-1}(\mathbb{W})(y)\}=\mathbb{F}^{-1}(\mathbb{G}^{-1}(W))(x).\end{aligned}$$

Hence the lemma. □

Lemma 2.19 *Let $\mathbb{F}:\mathbb{A}\to\mathbb{B}$ and $\mathbb{G}:\mathbb{B}\to\mathbb{C}$ be intuitionistic fuzzy proper functions. Then $(\mathbb{G}\circ\mathbb{F})(\mathbb{U})=\mathbb{G}(\mathbb{F}(\mathbb{U}))$, $\forall\mathbb{U}\ll\mathbb{A}$*

Proof. Let $\mathbb{U}\leq\mathbb{A}$ and $z\in Z$.

$$\begin{aligned}(\mathbb{G}\circ\mathbb{F})(\mathbb{U})(z) &= \bigsqcup_{x\in X}\{(\mathbb{G}\circ\mathbb{F})(x,z)\sqcap\mathbb{U}(x)\}\\ &= \bigsqcup_{x\in X}\left\{\bigsqcup_{y\in Y}\{\mathbb{F}(x,y)\sqcap\mathbb{G}(y,z)\}\sqcap\mathbb{U}(x)\right\}\\ &= \bigsqcup_{y\in Y}\left\{\bigsqcup_{x\in X}\{\mathbb{F}(x,y)\sqcap\mathbb{G}(y,z)\sqcap\mathbb{U}(x)\}\right\}\\ &= \bigsqcup_{y\in Y}\left\{\mathbb{G}(y,z)\sqcap\bigsqcup_{x\in X}\{\mathbb{F}(x,y)\sqcap\mathbb{U}(x)\}\right\}\\ &= \bigsqcup_{y\in Y}\{\mathbb{G}(y,z)\sqcap\mathbb{F}(\mathbb{U})(y)\}=\mathbb{G}(\mathbb{F}(U))(z).\end{aligned}$$

Hence the lemma. □

3 Fuzzy continuity on ISFTS

Definition 3.1 *Let X be nonempty set and $\mathbb{A}\in\zeta^X$. An intuitionistic smooth fuzzy topology $\mathcal{T}=(\mathcal{T}_1,\mathcal{T}_2)$ on an intuitionistic fuzzy set on $\mathbb{A}$ is a map $\mathcal{T}:\mathscr{I}_{\mathbb{A}}\to\zeta$ which satisfies the following properties:*

1. $\mathcal{T}(0,1)=\mathcal{T}(\mathbb{A})=(1,0)$,
2. $\mathcal{T}(\mathbb{U}\sqcap\mathbb{V})\gg\mathcal{T}(\mathbb{U})\sqcup\mathcal{T}(\mathbb{V})$,

3. $\mathcal{T}(\sqcup \mathbb{U}_i) \gg \sqcap \mathcal{T}(\mathbb{U}_i)$.

Then $(\mathbb{A}, \mathcal{T})$ *is said to be an intuitionistic smooth fuzzy topological space (ISFTS).*

Definition 3.2 *Let* $\mathbb{F} : (\mathbb{A}, \mathcal{T}) \to (\mathbb{B}, \mathcal{T}^{'})$ *and* $(\mathbb{A}, \mathcal{T})$, $(\mathbb{B}, \mathcal{T}^{'})$ *be ISFTSs.* $\mathbb{F}$ *is said to be intuitionistic smooth fuzzy continuous on* $\mathbb{A}$ *if* $\mathcal{T}(\mathbb{F}^{-1}(\mathbb{V})) \gg \mathcal{T}^{'}(\mathbb{V})$, $\forall \mathbb{V} \in \mathscr{I}_{\mathbb{B}}$.

Definition 3.3 $\mathbb{F} : (\mathbb{A}, \mathcal{T}) \to (\mathbb{B}, \mathcal{T}^{'})$ *is said to be intuitionistic weakly smooth fuzzy continuous on* $\mathbb{A}$*, if* $\mathcal{T}_1(\mathbb{F}^{-1}(\mathbb{V})) > 0$ *whenever* $\mathcal{T}_1^{'}(\mathbb{V}) > 0$ *and* $\mathcal{T}_2(\mathbb{F}^{-1}(\mathbb{V})) \leq \mathcal{T}_2^{'}(\mathbb{V})$, $\forall \mathbb{V} \in \mathscr{I}_{\mathbb{B}}$.

For our convenience, we denote $\zeta \setminus (0, 1)$ by ζ^*.

Definition 3.4 *Let* $(\mathbb{A}, \mathcal{T})$ *be an ISFTS and* $\mathbb{U} \in \mathscr{I}_{\mathbb{A}}$. $\mathbb{U}$ *is said to be a q-neighborhood of* $\mathbb{P}_x^{(r,s)} \in \mathbb{A}$ *if* $\mathcal{T}(\mathbb{U}) \in \zeta^*$ *and* $\mathbb{U} q \mathbb{P}_x^{(r,s)}[\mathbb{A}]$.

To discuss the point-wise continuity of a intuitionistic fuzzy proper function, in the context of ISFTSs, we introduce intuitionistic smooth fuzzy continuity at an IFP, intuitionistic weakly smooth fuzzy continuity at an IFP and intuitionistic qn-weakly smooth fuzzy continuity at an IFP as follow.

Definition 3.5 $\mathbb{F} : (\mathbb{A}, \mathcal{T}) \to (\mathbb{B}, \mathcal{T}^{'})$ *is said to be intuitionistic smooth fuzzy continuous at an IFP* $\mathbb{P}_x^{(r,s)} \in \mathbb{A}$ *if for every* $\mathbb{V} \ll \mathbb{B}$ *with* $\mathbb{F}(\mathbb{P}_x^{(r,s)}) \in \mathbb{V}$*, there exists* $\mathbb{U} \ll \mathbb{A}$ *such that* $\mathbb{P}_x^{(r,s)} \in \mathbb{U}$, $\mathbb{F}(\mathbb{U}) \ll \mathbb{V}$ *and* $\mathcal{T}(\mathbb{U}) \gg \mathcal{T}^{'}(\mathbb{V})$.

Definition 3.6 $\mathbb{F} : (\mathbb{A}, \mathcal{T}) \to (\mathbb{B}, \mathcal{T}^{'})$ *is said to be intuitionistic weakly smooth fuzzy continuous at an IFP* $\mathbb{P}_x^{(r,s)} \in \mathbb{A}$ *if for every* $\mathbb{V} \ll \mathbb{B}$ *with* $\mathcal{T}_1^{'}(\mathbb{V}) > 0$ *and* $\mathbb{F}(\mathbb{P}_x^{(r,s)}) \in \mathbb{V}$*, there exists* $\mathbb{U} \ll \mathbb{A}$ *such that* $\mathcal{T}_1(\mathbb{U}) > 0$, $\mathbb{P}_x^{(r,s)} \in \mathbb{U}$, $\mathbb{F}(\mathbb{U}) \ll \mathbb{V}$ *and* $\mathcal{T}_2(\mathbb{U}) \leq \mathcal{T}_2^{'}(\mathbb{V})$.

Definition 3.7 $\mathbb{F} : (\mathbb{A}, \mathcal{T}) \to (\mathbb{B}, \mathcal{T}^{'})$ *is said to be intuitionistic qn-weakly smooth fuzzy continuous at an IFP* $\mathbb{P}_x^{(r,s)} \in \mathbb{A}$ *if for every q-neighborhood* $\mathbb{V}$ *of* $\mathbb{F}(\mathbb{P}_x^{(r,s)}) \in \mathbb{V}$*, there exists a q-neighborhood* $\mathbb{U}$ *of* $\mathbb{P}_x^{(r,s)} \in \mathbb{A}$ *such that* $\mathbb{F}(\mathbb{U}) \ll \mathbb{V}$.

Theorem 3.8 *An intuitionistic fuzzy proper function* $\mathbb{F}$ *is intuitionistic smooth fuzzy continuous on* $\mathbb{A}$ *if and only if* $\mathbb{F}$ *is intuitionistic smooth fuzzy continuous at every IFP* $\mathbb{P}_x^{(r,s)} \in \mathbb{A}$.

Proof. Assume that $\mathbb{F} : \mathbb{A} \to \mathbb{B}$ is intuitionistic weakly smooth fuzzy continuous on $\mathbb{A}$. Let $\mathbb{P}_x^{(r,s)} \in \mathbb{A}$, $\mathbb{V} \ll \mathbb{B}$ with $\mathbb{F}(\mathbb{P}_x^{(r,s)}) = \mathbb{P}_y^{(r,s)} \in \mathbb{V}$, where y is such that $\mathbb{F}(x, y) = \mathbb{A}(x)$. If we take $\mathbb{U} = \mathbb{F}^{-1}(\mathbb{V})$, then we have $\mathcal{T}(\mathbb{U}) \gg \mathcal{T}^{'}(\mathbb{V})$ and $\mathbb{F}(\mathbb{U}) = \mathbb{F}(\mathbb{F}^{-1}(\mathbb{V})) \ll \mathbb{V}$. By Lemma 2.15 $\mathbb{P}_x^{(r,s)} \in \mathbb{U}$. Hence $\mathbb{F}$ is intuitionistic smooth fuzzy continuous at every IFP of $\mathbb{A}$. Conversely, assume that $\mathbb{F}$ is intuitionistic smooth fuzzy continuous at every IFP $\mathbb{P}_x^{(r,s)} \in \mathbb{A}$. Let $\mathbb{V} \ll \mathbb{B}$. For every $\mathbb{P}_x^{(r,s)} \in \mathbb{F}^{-1}(\mathbb{V})$, there exists $\mathbb{U}_{x;(r,s)} \ll \mathbb{A}$ such that $\mathbb{P}_x^{(r,s)} \in \mathbb{U}_{x;(r,s)}$, $\mathbb{F}(\mathbb{U}_{x;(r,s)}) \ll \mathbb{V}$, and $\mathcal{T}(\mathbb{U}_{x;(r,s)}) \gg \mathcal{T}^{'}(\mathbb{V})$. Since $\mathbb{F}^{-1}(\mathbb{V}) = \sqcup \mathbb{U}_{x;(r,s)}$, we get $\mathcal{T}(\mathbb{F}^{-1}(\mathbb{V})) = \mathcal{T}(\sqcup \mathbb{U}_{x;(r,s)}) \gg \sqcap \mathcal{T}(\mathbb{U}_{x;(r,s)}) \gg \mathcal{T}^{'}(\mathbb{V})$. Hence $\mathbb{F}$ is intuitionistic smooth fuzzy continuous on $\mathbb{A}$. □

Theorem 3.9 *An intuitionistic fuzzy proper function* $\mathbb{F}$ *is intuitionistic weakly smooth fuzzy continuous on* $\mathbb{A}$*, then* $\mathbb{F}$ *is intuitionistic weakly smooth fuzzy continuous at every IFP* $\mathbb{P}_x^{(r,s)} \in \mathbb{A}$.

Proof. Assume that $\mathbb{F}$ is intuitionistic weakly smooth fuzzy continuous on $\mathbb{A}$. Let $\mathbb{P}_x^{(r,s)} \in \mathbb{A}$, $\mathbb{V} \ll \mathbb{B}$ with $\mathcal{T}_1'(\mathbb{V}) > 0$ and $\mathbb{F}(\mathbb{P}_x^{(r,s)}) = \mathbb{P}_y^{(r,s)} \in \mathbb{V}$, where y is such that $\mathbb{F}(x,y) = \mathbb{A}(x)$. Let $\mathbb{U} = \mathbb{F}^{-1}(\mathbb{V})$. By our assumption $\mathcal{T}_1(\mathbb{U}) = \mathcal{T}_1(\mathbb{F}^{-1}(\mathbb{V})) > 0$ and $\mathcal{T}_2(\mathbb{U}) \ll \mathcal{T}_2'(\mathbb{V})$. By lemma 2.14 and 2.15, we have $\mathbb{F}(\mathbb{U}) \ll \mathbb{V}$ and $\mathbb{P}_x^{(r,s)} \in \mathbb{U}$. Hence $\mathbb{F}$ is intuitionistic weakly smooth fuzzy continuous at every $\mathbb{P}_x^{(r,s)} \in \mathbb{A}$. □

Counter example 3.10 *There exists $\mathbb{F} : (\mathbb{A}, \mathcal{T}) \to (\mathbb{B}, \mathcal{T}')$ such that it is intuitionistic weakly smooth fuzzy continuous at every IFP of $\mathbb{A}$ but not intuitionistic weakly smooth fuzzy continuous on $\mathbb{A}$.*

Let X and Y be set of all natural numbers and $\mathbb{A}^{[(0.8,0.2),(0.2,0.7),(0.2,0.7)...]}_{[1,2,3,...]}$, $\mathbb{B}^{[(0.9,0.1),(0.2,0.7),(0.2,0.7)...]}_{[1,2,3,...]}$, be intuitionistic fuzzy subsets of X and Y respectively. Define $\mathbb{U}_n \ll \mathbb{A}$, $\mathbb{V}_n \ll \mathbb{B}$ by $\mathbb{U}_n{}^{[(0.7-\frac{1}{n+1},0.3-\frac{1}{n+9}),(0.1,0.8),(0.1,0.8),...]}_{[1,2,3,...]}$, $\mathbb{V}_n{}^{[(0.7-\frac{1}{n+1},0.3-\frac{1}{n+9}),(0.1,0.8),(0.1,0.8),...]}_{[1,2,3,...]}$ $\forall n = 1,2,3.....$ If $\mathcal{T} = (\mathcal{T}_1, \mathcal{T}_2) : \mathscr{I}_{\mathbb{A}} \to \zeta$ by

$$\mathcal{T}_1(\mathbb{U}) = \begin{cases} 1, & \mathbb{U} = (0,1) \text{ or } \mathbb{A}, \\ \frac{1}{n+1}, & \mathbb{U} = \mathbb{U}_n, \forall n = 1,2,3,\ldots \\ 0, & \text{otherwise} . \end{cases}$$

$$\mathcal{T}_2(\mathbb{U}) = \begin{cases} 0, & \mathbb{U} = (0,1) \text{ or } \mathbb{A}, \\ \frac{1}{n+1}, & \mathbb{U} = \mathbb{U}_n, \forall n = 1,2,3,\ldots \\ 1, & \text{otherwise} . \end{cases}$$

and $\mathcal{T}' = (\mathcal{T}_1', \mathcal{T}_2') : \mathscr{I}_{\mathbb{B}} \to \zeta$ by

$$\mathcal{T}_1'(\mathbb{V}) = \begin{cases} 1, & \mathbb{V} = (0,1) \text{ or } \mathbb{B}, \\ 0.5, & \mathbb{V} = \mathbb{V}_n, \forall n = 1,2,3,\ldots \\ 0.5, & \mathbb{V} = \vee \mathbb{V}_n \\ 0, & \text{otherwise}, \end{cases}$$

$$\mathcal{T}_2'(\mathbb{V}) = \begin{cases} 0, & \mathbb{V} = (0,1) \text{ or } \mathbb{B}, \\ \frac{1}{n+1}, & \mathbb{V} = \mathbb{V}_n, \forall n = 1,2,3,\ldots \\ 1, & \text{otherwise}, \end{cases}$$

then obviously $(\mathbb{A}, \mathcal{T})$, $(\mathbb{B}, \mathcal{T}')$ are intuitionistic smooth fuzzy topological spaces.
Let the intuitionistic fuzzy proper function $\mathbb{F} : (\mathbb{A}, \mathcal{T}) \to (\mathbb{B}, \mathcal{T}')$ be defined by $\mathbb{F}(k,j) = \mathbb{A}(k)$, when $j = k$, $\mathbb{F}(k,j) = (0,1)$, when $j \neq k$. We first note that $\mathbb{F}$ is not intuitionistic weakly smooth fuzzy continuous on $\mathbb{A}$. Let $\mathbb{V} = \sqcup \mathbb{V}_n$ with $\mathcal{T}_1'(\mathbb{V}) = 0.5$. If $x \in X$ and $y \in Y$ such

that $\mathbb{F}(x,y)=\mathbb{A}(x)$, then

$$\begin{aligned}\mathbb{F}^{-1}(\mathbb{V})(x) &= \mathbb{F}^{-1}(\sqcup\mathbb{V}_n)=\mathbb{A}(x)\sqcap(\sqcup\mathbb{V}_n)(y)\\ &= (\mu_{\mathbb{A}}(x),\nu_{\mathbb{A}}(x))\sqcap(\vee\mu_{\mathbb{V}_n}(y),\wedge\nu_{\mathbb{V}_n}(y))\\ &= (\mu_{\mathbb{A}}(x)\wedge(\vee\mu_{\mathbb{V}_n})(y),\nu_{\mathbb{A}}(x)\vee(\wedge\nu_{\mathbb{V}_n})(y))\\ &= (\vee(\mu_{\mathbb{A}}(x)\wedge\mu_{\mathbb{V}_n})(y),\wedge(\nu_{\mathbb{A}}(x)\vee\nu_{\mathbb{V}_n})(y))\\ &= \sqcup(\mu_{\mathbb{A}}(x)\wedge\mu_{\mathbb{V}_n}(y),\nu_{\mathbb{A}}(x)\vee\nu_{\mathbb{V}_n}(y))\\ &= \sqcup(\mathbb{A}(x)\sqcap\mathbb{V}_n(y))\\ &= \sqcup\mathbb{F}^{-1}(\mathbb{V}_n)(x)=\sqcup\mathbb{U}_n(x).\end{aligned}$$

Hence

$$\mathcal{T}_1(\mathbb{F}^{-1}(\mathbb{V}))=\mathcal{T}_1(\vee\mathbb{U}_n)\geq\wedge\mathcal{T}_1(\mathbb{U}_n)=\wedge\frac{1}{n+1}=0.$$

Next we prove that $\mathbb{F}$ is intuitionistic weakly smooth fuzzy continuous at every IFP in $\mathbb{A}$. Fix $\mathbb{P}_k^{(r,s)}\in\mathbb{A}$ be arbitrary. $\mathbb{F}(\mathbb{P}_k^{(r,s)})=\mathbb{P}_k^{(r,s)}$, for all $\mathbb{P}_k^{(r,s)}\in\mathbb{A}$ and $\mathbb{V}_n(j)=\mathbb{U}_n(j)$, $\forall j,n\in\mathbb{N}$. If $\mathbb{F}(\mathbb{P}_k^{(r,s)})\in\mathbb{V}_n$ with $\mathcal{T}_1^{'}(V_n)>0$, then $\mathbb{P}_k^{(r,s)}\in\mathbb{U}_n$ and $\mathbb{F}(\mathbb{U}_n)\ll\mathbb{V}_n$, $\mathcal{T}_1(\mathbb{U}_n)>0$ and $\mathcal{T}_2(\mathbb{U}_n)\leq\mathcal{T}_2^{'}(\mathbb{V}_n)$.

The converse of Theorem 3.9 can be achieved under some sufficient conditions, for which it is necessary to introduce the following definitions.

Definition 3.11 *Let $\mathbb{F}:(\mathbb{A},\mathcal{T})\to(\mathbb{B},\mathcal{T}^{'})$ and $\alpha\in(0,1]$. $\mathbb{F}$ is said to be α-weakly intuitionistic smooth fuzzy continuous on $\mathbb{A}$, if $\mathcal{T}_1(\mathbb{F}^{-1}(\mathbb{V}))\geq\alpha$ whenever $\mathbb{V}\ll B$ and $\mathcal{T}_1^{'}(\mathbb{V})\geq\alpha$ and $\mathcal{T}_2(\mathbb{F}^{-1}(\mathbb{V}))\leq\mathcal{T}_2^{'}(\mathbb{V})$.*

Definition 3.12 *Let $\mathbb{F}:(\mathbb{A},\mathcal{T})\to(\mathbb{B},\mathcal{T}^{'})$ and $\alpha\in(0,1]$. $\mathbb{F}$ is said to be α-weakly intuitionistic smooth fuzzy continuous at a fuzzy point $\mathbb{P}_x^{(r,s)}\in\mathbb{A}$ if for every $\mathbb{V}\ll\mathbb{B}$ with $\mathcal{T}_1^{'}(\mathbb{V})\geq\alpha$ and $\mathbb{F}(\mathbb{P}_x^{(r,s)})\in\mathbb{V}$, there exists $\mathbb{U}\ll\mathbb{A}$ with $\mathcal{T}_1(\mathbb{U})\geq\alpha$ and $\mathbb{P}_x^{(r,s)}\in\mathbb{U}$ such that $\mathbb{F}(\mathbb{U})\ll\mathbb{V}$ and $\mathcal{T}_2(\mathbb{U})\leq\mathcal{T}_2^{'}(\mathbb{V})$.*

Definition 3.13 *An intuitionistic smooth fuzzy topological space $(\mathbb{A},\mathcal{T})$ is said to be positive minimum intuitionistic smooth topology if $\bigwedge_{i\in\Gamma}\mathcal{T}_1(\mathbb{U}_i)>0$, whenever $\mathbb{U}_i\in\mathscr{I}_{\mathbb{A}}$ and $\mathcal{T}_1(\mathbb{U}_i)>0$ for all $i\in\Gamma$.*

Theorem 3.14 *An intuitionistic fuzzy proper function $\mathbb{F}$ is α-weakly intuitionistic smooth fuzzy continuous on $\mathbb{A}$ if and only if $\mathbb{F}$ is α-weakly intuitionistic smooth fuzzy continuous at every fuzzy point $\mathbb{P}_x^{(r,s)}\in\mathbb{A}$.*

Proof. Assume that $\mathbb{F}$ is α-weakly intuitionistic smooth fuzzy continuous on $\mathbb{A}$. Let $\mathbb{P}_x^{(r,s)}\in\mathbb{A}$, $\mathbb{V}\ll\mathbb{B}$ with $\mathcal{T}_1^{'}(\mathbb{V})\geq\alpha$ and $\mathbb{F}(\mathbb{P}_x^{(r,s)})=\mathbb{P}_y^{(r,s)}\in\mathbb{V}$, where y is such that $\mathbb{F}(x,y)=\mathbb{A}(x)$. If we take $\mathbb{U}=\mathbb{F}^{-1}(\mathbb{V})$, then $\mathcal{T}_1(\mathbb{U})=\mathcal{T}_1(\mathbb{F}^{-1}(\mathbb{V}))\geq\alpha$, $\mathbb{F}(\mathbb{U})\ll\mathbb{V}$ and $\mathbb{P}_x^{(r,s)}\in\mathbb{U}$. Hence $\mathbb{F}$ is intuitionistic α -weakly smooth fuzzy continuous at every $\mathbb{P}_x^{(r,s)}\in\mathbb{A}$. Conversely, assume that $\mathbb{F}$ is α-weakly intuitionistic smooth fuzzy continuous at every IFP $\mathbb{P}_x^{(r,s)}\in\mathbb{A}$. Let $\mathbb{V}\ll\mathbb{B}$, For every $\mathbb{P}_x^{(r,s)}\in\mathbb{F}^{-1}(\mathbb{V})$, there exists $\mathbb{U}_{x;(r,s)}\ll\mathbb{A}$ such that $\mathbb{P}_x^{(r,s)}\in\mathbb{U}_{x;(r,s)}$, $\mathbb{F}(\mathbb{U}_{x;(r,s)})\ll\mathbb{V}$, and $\mathcal{T}_1(\mathbb{U}_{x;(r,s)})\geq\alpha$. Since $\mathbb{F}^{-1}(\mathbb{V})=\sqcup\mathbb{U}_{x;(r,s)}$, we have $\mathcal{T}_2(\mathbb{F}^{-1}(\mathbb{V}))=\mathcal{T}_2(\sqcup\mathbb{U}_{x;(r,s)})\leq\vee\mathcal{T}_2(\mathbb{U}_{x;(r,s)})$ $\leq\mathcal{T}_2^{'}(\mathbb{V})$ we also have, $\mathcal{T}_1(\mathbb{F}^{-1}(\mathbb{V}))=\mathcal{T}_1(\sqcup\mathbb{U}_{x;(r,s)})\geq\wedge\mathcal{T}_1(\mathbb{U}_{x;(r,s)})\geq\alpha$. Hence $\mathbb{F}$ is α-weakly intuitionistic smooth fuzzy continuous on $\mathbb{A}$. □

Similarly, we can prove the following theorem.

Theorem 3.15 *Let $\mathbb{F} : (\mathbb{A}, \mathcal{T}) \to (\mathbb{B}, \mathcal{T}')$ and $(\mathbb{A}, \mathcal{T})$ be a positive minimum intuitionistic smooth fuzzy topological space. $\mathbb{F}$ is intuitionistic weakly smooth fuzzy continuous on $\mathbb{A}$ if and only if $\mathbb{F}$ is intuitionistic weakly smooth fuzzy continuous at every IFP of $\mathbb{A}$.*

Theorem 3.16 *If an intuitionistic fuzzy proper function $\mathbb{F} : (\mathbb{A}, \mathcal{T}) \to (\mathbb{B}, \mathcal{T}')$ is intuitionistic weakly smooth fuzzy continuous on $\mathbb{A}$, then $\mathbb{F}$ is qn-weakly intuitionistic smooth fuzzy continuous at every IFP in $\mathbb{A}$.*

Proof. Assume $\mathbb{F}$ is intuitionistic weakly smooth fuzzy continuous on $\mathbb{A}$. Let $\mathbb{V} \leq \mathbb{B}$ be a q-neighborhood of $\mathbb{F}(\mathbb{P}_x^{(r,s)}) = \mathbb{P}_y^{(r,s)}$, where $y \in Y$ is such that $\mathbb{F}(x, y) = \mathbb{A}(x)$. Then we have $\mathcal{T}_1'(\mathbb{V}) > 0$ and $\mathbb{P}_y^{(r,s)} q\mathbb{V}[\mathbb{B}]$. Take $\mathbb{U} = \mathbb{F}^{-1}(\mathbb{V})$. By our assumption $\mathcal{T}_1(\mathbb{U}) > 0$, $\mathcal{T}_2(\mathbb{U}) \leq \mathcal{T}_2'(\mathbb{V})$ and $\mathbb{F}(\mathbb{U}) \ll \mathbb{V}$. Clearly by Lemma 2.16, we have $\mathbb{P}_x^{(r,s)} q\mathbb{U}[\mathbb{A}]$. Hence $\mathbb{F}$ is qn-weakly intuitionistic smooth fuzzy continuous at every IFP of $\mathbb{A}$. $\square$

Counter example 3.17 *There exists an intuitionistic fuzzy proper function on $\mathbb{A}$ such that it is qn-weakly intuitionistic smooth fuzzy continuous at every IFP of $\mathbb{A}$ but it is not intuitionistic weakly smooth fuzzy continuous on $\mathbb{A}$.*

Let $X = \{x, y\}$, $Y = \{a, b\}$, $\mathbb{A}_{[x,y]}^{[(1,0),(0.2,0.2)]} \in \zeta^X$, $\mathbb{B}_{[a,b]}^{[(1,0),(0.9,0.1)]} \in \zeta^Y$, ${\mathbb{U}_1}_{[x,y]}^{[(0.8,0.1),(0.1,0.2)]}$ and ${\mathbb{V}_1}_{[a,b]}^{[(0.8,0.1),(0.8,0.2)]}$. We define $\mathcal{T} = (\mathcal{T}_1, \mathcal{T}_2) : \mathscr{I}_{\mathbb{A}} \to \zeta$ by

$$\mathcal{T}_1(\mathbb{U}) = \begin{cases} 1, & \mathbb{U} = (0,1) \text{ or } \mathbb{A}, \\ 0.9, & \mathbb{U} = \mathbb{U}_1, \\ 0, & \text{otherwise} , \end{cases}$$

$$\mathcal{T}_2(\mathbb{U}) = \begin{cases} 0, & \mathbb{U} = (0,1) \text{ or } \mathbb{A}, \\ 0.1, & \mathbb{U} = \mathbb{U}_1, \\ 1, & \text{otherwise} , \end{cases}$$

and $\mathcal{T}' = (\mathcal{T}_1', \mathcal{T}_2') : \mathscr{I}_{\mathbb{B}} \to \zeta$ by

$$\mathcal{T}_1'(\mathbb{V}) = \begin{cases} 1, & \mathbb{V} = (0,1) \text{ or } \mathbb{B}, \\ 0.8, & \mathbb{V} = \mathbb{V}_1, \\ 0, & \text{otherwise} . \end{cases}$$

$$\mathcal{T}_2'(\mathbb{V}) = \begin{cases} 0, & \mathbb{V} = (0,1) \text{ or } \mathbb{B}, \\ 0.2, & \mathbb{V} = \mathbb{V}_1, \\ 1, & \text{otherwise} . \end{cases}$$

Define a fuzzy proper function $\mathbb{F} : (\mathbb{A}, \mathcal{T}_1) \to (\mathbb{B}, \mathcal{T}_1')$ by

$$\mathbb{F}(x, a) = \mathbb{A}(x), \mathbb{F}(x, b) = (0, 1), \mathbb{F}(y, a) = (0, 1), \mathbb{F}(y, b) = \mathbb{A}(y).$$

$\mathbb{F}$ is not weakly intuitionistic smooth fuzzy continuous on $\mathbb{A}$, since $\mathcal{T}_1'\left({\mathbb{V}_1}_{[a,b]}^{[(0.8,0.1),(0.8,0.2)]}\right) = 0.8 > 0$. Now

$$\mathbb{F}^{-1}(\mathbb{V})(x) = \mathbb{A}(x) \sqcap \mathbb{V}(a) = (1, 0) \sqcap (0.8, 0.1) = (0.8, 0.1)$$

$$\mathbb{F}^{-1}(\mathbb{V})(y) = \mathbb{A}(y) \sqcap \mathbb{V}(b) = (0.2, 0.2) \sqcap (0.8, 0.2) = (0.2, 0.2)$$

Hence $\mathcal{T}_1 \left(\mathbb{F}^{-1}(\mathbb{V}_1)_{[x,y]}^{[(0.8,0.1),(0.2,0.2)]} \right) = 0$.

We claim that $\mathbb{F}$ is qn-weakly intuitionistic smooth fuzzy continuous at every IFP of $\mathbb{A}$. We note that if $\mathbb{P}_x^{(r,s)} \in \mathbb{A}$, then $\mathbb{F}(\mathbb{P}_x^{(r,s)}) = \mathbb{P}_a^{(r,s)}$.

Case 1: $0 \le s < 0.8$ and $0 < r \le 1$.

$$\mu_{\mathbb{B}}(a) \wedge s = 1 \wedge s = s < 0.8 = 0.8 \vee 0 = \mu_{\mathbb{V}_1}(a) \vee \nu_{\mathbb{B}}(a).$$

In this case, $\mathbb{V}_1$ and $\mathbb{B}$ are the possible q-neighborhoods of $\mathbb{P}_a^{(r,s)}$ in $\mathbb{B}$. For $\mathbb{V}_1$, we find that $\mathbb{U}_1$ is q-neighborhood of $\mathbb{P}_x^{(r,s)}$ and $\mathbb{F}(\mathbb{U}_1) \ll \mathbb{V}_1$. Since

$$\mu_{\mathbb{A}}(x) \wedge s = 1 \wedge s = s < 0.8 = 0.8 \vee 0 = \mu_{\mathbb{U}_1}(x) \vee \nu_{\mathbb{A}}(x).$$

Hence $\mathbb{U}_1$ is a q-neighborhood $\mathbb{P}_x^{(r,s)}$.

$$\begin{aligned} \mathbb{F}(\mathbb{U}_1)(a) &= \sqcup \{\mathbb{F}(x,a) \sqcap \mathbb{U}_1(x), \mathbb{F}(y,a) \sqcap \mathbb{U}_1(y)\} \\ &= (1,0) \sqcap (0.8, 0.1) = (0.8, 0.1) \\ \mathbb{F}(\mathbb{U}_1)(b) &= \sqcup \{\mathbb{F}(x,b) \sqcap \mathbb{U}_1(x), \mathbb{F}(y,b) \sqcap \mathbb{U}_1(y)\} \\ &= (0.2, 0.2) \sqcap (0.1, 0.2) = (0.1, 0.2). \end{aligned}$$

Therefore $\mathbb{F}(\mathbb{U}_1)_{[a,b]}^{[(0.8,0.1),(0.1,0.2)]} \ll {\mathbb{V}_1}_{[a,b]}^{[(0.8,0.1),(0.7,0.2)]}$.

For $\mathbb{B}$, we choose $\mathbb{A}$ which is a q-neighborhood of $\mathbb{P}_x^{(r,s)}$ such that $\mathbb{F}(\mathbb{A}) \ll \mathbb{B}$.

Case 2: $0.8 \le s < 1$ and $0.1 < r \le 1$.

$$\nu_{\mathbb{B}}(a) \vee r = 0 \vee r = r > 0.1 = 0.1 \wedge 1 = \nu_{\mathbb{V}_1}(a) \wedge \mu_{\mathbb{B}}(a).$$

In this case, $\mathbb{V}_1$ and $\mathbb{B}$ are the possible q-neighborhoods of $\mathbb{P}_a^{(r,s)}$ in $\mathbb{B}$. For $\mathbb{V}_1$, we find $\mathbb{U}_1$ is q-neighborhood of $\mathbb{P}_x^{(r,s)}$ and $\mathbb{F}(\mathbb{U}_1) \ll \mathbb{V}_1$. Since

$$\nu_{\mathbb{A}}(x) \vee r = 0 \vee r = r > 0.1 = 0.1 \wedge 1 = \nu_{\mathbb{U}_1}(x) \wedge \mu_{\mathbb{A}}(x),$$

$\mathbb{U}_1$ is q-neighborhood $\mathbb{P}_x^{(r,s)}$. Clearly $\mathbb{F}(\mathbb{U}_1) \ll \mathbb{V}_1$. For $\mathbb{B}$, we choose $\mathbb{A}$ as before.

Case 3: $0.8 \le s < 1$ and $0 < r \le 0.1$

In this case B is the only q-neighborhood of $\mathbb{P}_a^{(r,s)}$. For $\mathbb{B}$, we choose $\mathbb{A}$ as before.

We note that if $\mathbb{P}_y^{(r,s)} \in \mathbb{A}$, then $\mathbb{F}(\mathbb{P}_y^{(r,s)}) = \mathbb{P}_b^{(r,s)}$.

Case 1: $0.8 \le s < 1$ and $0 < r \le 0.1$.

$$\nu_{\mathbb{B}}(b) \vee r = 0.1 \vee r > 0.1 \vee 0.2 = 0.2 = 0.2 \wedge 0.9 = \nu_{\mathbb{V}_1}(b) \wedge \mu_{\mathbb{B}}(b).$$

In this case, $\mathbb{V}_1$ and $\mathbb{B}$ are the possible q-neighborhoods of $\mathbb{P}_b^{(r,s)}$ in $\mathbb{B}$. For $\mathbb{V}_1$, we find $\mathbb{U}_1$ is q-neighborhood of $\mathbb{P}_y^{(r,s)}$ and $\mathbb{F}(\mathbb{U}_1) \ll \mathbb{V}_1$. Since

$$\nu_{\mathbb{A}}(y) \vee r = 0.2 \vee r > 0.1 \vee 0.2 = 0.2 = 0.2 \wedge 0.2 = \nu_{\mathbb{U}_1}(y) \vee \mu_{\mathbb{A}}(y),$$

$\mathbb{U}_1$ is q-neighborhood $\mathbb{P}_x^{(r,s)}$ and $\mathbb{F}(\mathbb{U}_1)_{[a,b]}^{[(0.8,0.1),(0.1,0.2)]} \ll {\mathbb{V}_1}_{[a,b]}^{[(0.8,0.1),(0.7,0.2)]}$.

For $\mathbb{B}$, we choose $\mathbb{A}$ which is q-neighborhood of $\mathbb{P}_y^{(r,s)}$ such that $\mathbb{F}(\mathbb{A}) \ll \mathbb{B}$.

Case 2: $0 \le s < 0.8$ and $0 < r \le o.2$.

$$\mu_{\mathbb{B}}(b) \wedge s = 0.9 \wedge s < 0.9 \wedge 0.8 = 0.8 = 0.8 \vee 0.1 = \mu_{\mathbb{V}_1}(b) \vee \nu_{\mathbb{B}}(b).$$

In this case, $\mathbb{V}_1$ and $\mathbb{B}$ are the possible q-neighborhoods of $\mathbb{P}_b^{(r,s)}$ in $\mathbb{B}$. For $\mathbb{V}_1$, we find $\mathbb{U}_1$ is q-neighborhood of $\mathbb{P}_y^{(r,s)}$ and $\mathbb{F}(\mathbb{U}_1) \ll \mathbb{V}_1$. Since

$$\nu_{\mathbb{A}}(y) \vee r = 0.2 \vee r > 0.2 \vee 0 = 0.2 = 0.2 \wedge 0.2 = \nu_{\mathbb{U}_1}(y) \wedge \mu_{\mathbb{A}}(y),$$

$\mathbb{U}_1$ is q-neighborhood $\mathbb{P}_y^{(r,s)}$. Clearly $\mathbb{F}(\mathbb{U}_1) \ll \mathbb{V}_1$. For $\mathbb{B}$, we choose $\mathbb{A}$ as before.
Case 3: $0.8 \leq s < 1$ and $0 < r \leq 0.2$
In this case B is the only q-neighborhood of $\mathbb{P}_a^{(r,s)}$. For $\mathbb{B}$, we choose $\mathbb{A}$ as before. Hence $\mathbb{F}$ is qn-weakly smooth fuzzy continuous at every IFP $\mathbb{A}$.

Theorem 3.18 *If $\mathbb{F}$ is α-weakly intuitionistic smooth fuzzy continuous $\forall \alpha \in (0,1]$ on $\mathbb{A}$, then $\mathbb{F}$ is intuitionistic weakly smooth fuzzy continuous on $\mathbb{A}$.*

Proof. Let $\mathbb{V} \ll \mathbb{B}$ with $\mathcal{T}_1'(\mathbb{V}) > 0$. Choose α such that $0 < \alpha \leq \mathcal{T}_1'(\mathbb{V})$. Since $\mathbb{F}$ is α-weakly smooth fuzzy continuous, $\mathcal{T}_1(\mathbb{F}^{-1}(\mathbb{V})) \geq \alpha > 0$ and $\mathcal{T}_2(\mathbb{F}^{-1}(\mathbb{V})) \leq \mathcal{T}_2'(\mathbb{V})$. Hence $\mathbb{F}$ is intuitionistic weakly smooth fuzzy continuous on $\mathbb{A}$. □

Counter example 3.19 *There exists an intuitionistic fuzzy proper function $\mathbb{F}$ which is weakly smooth fuzzy continuous but not α-weakly smooth fuzzy continuous for some $\alpha \in (0,1]$.*

Let $X = \{x, y\}$, $Y = \{a, b\}$. If $\mathbb{A}_{[r,s]}^{[(0.7,0.3),(0.5,0.4)]}$, $\mathbb{B}_{[a,b]}^{[(0.8,0.2),(0.6,0.4)]}$, ${\mathbb{U}_1}_{[r,s]}^{[(0.6,0.4),(0.5,0.4)]}$, ${\mathbb{V}_1}_{[a,b]}^{[(0.6,0.4),(0.6,0.4)]}$, then $\mathbb{U}_1 \ll \mathbb{A} \in \zeta^X$, $\mathbb{V}_1 \ll \mathbb{B} \in \zeta^Y$. We define smooth fuzzy topologies $\mathcal{T} = (\mathcal{T}_1, \mathcal{T}_2)$ on $\mathscr{I}_{\mathbb{A}}$ and $\mathcal{T}' = (\mathcal{T}_1', \mathcal{T}_2')$ on $\mathscr{I}_{\mathbb{B}}$ by

$$\mathcal{T}_1(\mathbb{U}) = \begin{cases} 1, & \mathbb{U} = (0,1) \text{ or } \mathbb{A}, \\ 0.6, & \mathbb{U} = \mathbb{U}_1, \\ 0, & \text{otherwise} \end{cases}$$

$$\mathcal{T}_2(\mathbb{U}) = \begin{cases} 0, & \mathbb{U} = (0,1) \text{ or } \mathbb{A}, \\ 0.2, & \mathbb{U} = \mathbb{U}_1, \\ 1, & \text{otherwise} \end{cases}$$

and

$$\mathcal{T}_1'(\mathbb{V}) = \begin{cases} 1, & \mathbb{V} = (0,1) \text{ or } \mathbb{B}, \\ 0.7, & \mathbb{V} = \mathbb{V}_1, \\ 0, & \text{otherwise }. \end{cases}$$

$$\mathcal{T}_2'(\mathbb{V}) = \begin{cases} 0, & \mathbb{V} = (0,1) \text{ or } \mathbb{B}, \\ 0.3, & \mathbb{V} = \mathbb{V}_1, \\ 1, & \text{otherwise }. \end{cases}$$

It is clear that $(\mathbb{A}, \mathcal{T})$, $(\mathbb{B}, \mathcal{T}')$ are two intuitionistic smooth fuzzy topological spaces. Let the proper function $\mathbb{F} : (\mathbb{A}, \mathcal{T}) \to (\mathbb{B}, \mathcal{T}')$ be defined by

$$\mathbb{F}(x, a) = \mathbb{A}(x), \mathbb{F}(x, b) = (0,1), \mathbb{F}(y, a) = (0,1), \mathbb{F}(y, b) = \mathbb{A}(y).$$

We note that $\mathbb{F}^{-1}(\mathbb{V}_1)_{[x,y]}^{[(0.6,0.4),(0.5,0.4)]}$ and $\mathbb{F}^{-1}(\mathbb{B})_{[x,y]}^{[(0.7,0.3),(0.5,0.4)]}$. Therefore $\mathbb{F}$ is intuitionistic weakly smooth fuzzy continuous on $\mathbb{A}$. But $\mathbb{F}$ is not α-weakly smooth fuzzy continuous for $\alpha = 0.65$. Because $\mathcal{T}_1'(\mathbb{V}_1) = 0.7 > \alpha$ but $\mathcal{T}_1(\mathbb{F}^{-1}(\mathbb{V}_1)) = \mathcal{T}_1(\mathbb{U}_1) = 0.6 \not> \alpha$ and $\mathcal{T}_2(\mathbb{F}^{-1}(\mathbb{V}_1)) = 0.2 \leq 0.3 = \mathcal{T}_2'(\mathbb{V}_1)$

Definition 3.20 *Let* $\mathbb{F} : (\mathbb{A}, \mathcal{T}) \to (\mathbb{B}, \mathcal{T}^{'})$. $\mathbb{F}$ *is said to be* $(0, 1)$*-weakly intuitionistic smooth fuzzy continuous on* $\mathbb{A}$*, if* $\mathcal{T}_1(\mathbb{F}^{-1}(\mathbb{V})) > 0$ *whenever* $\mathcal{T}_1^{'}(\mathbb{V}) > 0$.

Definition 3.21 *Let* $\mathbb{F} : (\mathbb{A}, \mathcal{T}) \to (\mathbb{B}, \mathcal{T}^{'})$. $\mathbb{F}$ *is said to be* $(0, 1)$*- weakly intuitionistic smooth fuzzy continuous at an IFP* $\mathbb{P}_x^{(r,s)} \in \mathbb{A}$ *if for every* $\mathbb{V} \ll \mathbb{B}$ *with* $\mathcal{T}_1^{'}(\mathbb{V}) > 0$ *and* $\mathbb{F}(\mathbb{P}_x^{(r,s)}) \in \mathbb{V}$*, there exists* $\mathbb{U} \ll \mathbb{A}$ *such that* $\mathcal{T}_1(\mathbb{U}) > 0$*,* $\mathbb{P}_x^{(r,s)} \in \mathbb{U}$ *and* $\mathbb{F}(\mathbb{U}) \ll \mathbb{V}$.

We point out that as in the case of intuitionistic weakly smooth fuzzy continuous function, if $\mathbb{F}$ is $(0, 1)$-weakly intuitionistic smooth fuzzy continuous on $\mathbb{A}$, then $\mathbb{F}$ is $(0, 1)$-weakly intuitionistic smooth fuzzy continuous at every IFP $\mathbb{P}_x^{(r,s)} \in \mathbb{A}$. But the converse of this statement is not true.

Definition 3.22 *Let* $\mathbb{F} : (\mathbb{A}, \mathcal{T}) \to (\mathbb{B}, \mathcal{T}^{'})$ *and* $(\alpha, \beta) \in \zeta^*$. $\mathbb{F}$ *is said to be* (α, β)*-weakly intuitionistic smooth fuzzy continuous on* $\mathbb{A}$*, if* $\mathcal{T}(\mathbb{F}^{-1}(\mathbb{V})) \gg (\alpha, \beta)$ *whenever* $\mathcal{T}^{'}(\mathbb{V}) \gg (\alpha, \beta)$.

Definition 3.23 (Cf. [17]) *Let* $\mathbb{F} : (\mathbb{A}, \mathcal{T}) \to (\mathbb{B}, \mathcal{T}^{'})$ *and* $(\alpha, \beta) \in \zeta^*$. $\mathbb{F}$ *is said to be* (α, β)*- weakly intuitionistic smooth fuzzy continuous at an IFP* $\mathbb{P}_x^{(r,s)} \in \mathbb{A}$ *if for every* $\mathbb{V} \ll \mathbb{B}$ *such that* $\mathcal{T}^{'}(\mathbb{V}) \gg (\alpha, \beta)$*,* $\mathbb{F}(\mathbb{P}_x^{(r,s)}) \in \mathbb{V}$*, there exists* $\mathbb{U} \ll \mathbb{A}$ *such that* $\mathcal{T}(\mathbb{U}) \gg (\alpha, \beta)$*,* $\mathbb{P}_x^{(r,s)} \in \mathbb{U}$ *and* $\mathbb{F}(\mathbb{U}) \ll \mathbb{V}$.

Theorem 3.24 *An intuitionistic fuzzy proper function* $\mathbb{F}$ *is* (α, β)*-weakly intuitionistic smooth fuzzy continuous on* $\mathbb{A}$ *if and only if* $\mathbb{F}$ *is* (α, β)*-weakly intuitionistic smooth fuzzy continuous at every fuzzy point* $\mathbb{P}_x^{(r,s)} \in \mathbb{A}$.

Proof. Assume $\mathbb{F}$ is (α, β)-weakly intuitionistic smooth fuzzy continuous on $\mathbb{A}$. Let $\mathbb{P}_x^{(r,s)} \in \mathbb{A}$, $\mathbb{V} \ll \mathbb{B}$ with $\mathcal{T}_1^{'}(\mathbb{V}) \geq \alpha$, $\mathcal{T}_2^{'}(\mathbb{V}) \leq \beta$ and $\mathbb{F}(\mathbb{P}_x^{(r,s)}) = \mathbb{P}_y^{(r,s)} \in \mathbb{V}$, where y is such that $\mathbb{F}(x, y) = \mathbb{A}(x)$. If we take $\mathbb{U} = \mathbb{F}^{-1}(\mathbb{V})$, then $\mathcal{T}_1(\mathbb{U}) = \mathcal{T}_1(\mathbb{F}^{-1}(\mathbb{V})) \geq \alpha$, $\mathcal{T}_2(\mathbb{U}) = \mathcal{T}_2(\mathbb{F}^{-1}(\mathbb{V})) \leq \beta$, $\mathbb{F}(\mathbb{U}) \leq \mathbb{V}$ and $\mathbb{P}_x^{(r,s)} \in \mathbb{U}$. Hence $\mathbb{F}$ is intuitionistic (α, β) -weakly smooth fuzzy continuous at every $\mathbb{P}_x^{(r,s)} \in \mathbb{A}$. Conversely, assume that $\mathbb{F}$ is (α, β)-weakly intuitionistic smooth fuzzy continuous at every IFP $\mathbb{P}_x^{(r,s)} \in \mathbb{A}$. Let $\mathbb{V} \ll \mathbb{B}$,For every $\mathbb{P}_x^{(r,s)} \in \mathbb{F}^{-1}(\mathbb{V})$, there exists $\mathbb{U}_{x;(r,s)} \ll \mathbb{A}$ such that $\mathbb{P}_x^{(r,s)} \in \mathbb{U}_{x;(r,s)}$, $\mathbb{F}(\mathbb{U}_{x;(r,s)}) \ll \mathbb{V}$, and $\mathcal{T}_1(\mathbb{U}_{x;(r,s)}) \geq \alpha$, $\mathcal{T}_2(\mathbb{U}_{x;(r,s)}) \leq \beta$. Then clearly $\mathbb{F}^{-1}(\mathbb{V}) = \sqcup \mathbb{U}_{x;(r,s)}$, $\mathcal{T}_1(\mathbb{F}^{-1}(\mathbb{V})) = \mathcal{T}_1(\sqcup \mathbb{U}_{x;(r,s)}) \geq \wedge \mathcal{T}_1(\mathbb{U}_{x;(r,s)}) \geq \alpha$ and $\mathcal{T}_2(\mathbb{F}^{-1}(\mathbb{V})) = \mathcal{T}_2(\sqcup \mathbb{U}_{x;(r,s)}) \leq \vee \mathcal{T}_2(\mathbb{U}_{x;(r,s)}) \leq \beta$. Hence $\mathbb{F}$ is (α, β)-weakly intuitionistic smooth fuzzy continuous on $\mathbb{A}$. □

Theorem 3.25 *If* $\mathbb{F}$ *is* (α, β)*-weakly intuitionistic smooth fuzzy continuous* $\forall (\alpha, \beta) \in \zeta^*$ *on* $\mathbb{A}$*, then* $\mathbb{F}$ *is intuitionistic* $(0, 1)$*-weakly smooth fuzzy continuous on* $\mathbb{A}$.

Proof. Let $\mathbb{V} \leq \mathbb{B}$ with $\mathcal{T}_1^{'}(\mathbb{V}) > 0$. Choose α, β such that $0 < \alpha \leq \mathcal{T}_1^{'}(\mathbb{V})$ and $\mathcal{T}_2^{'}(\mathbb{V}) \leq \beta < 1$. Since $\mathbb{F}$ is (α, β)-weakly smooth fuzzy continuous, $\mathcal{T}_1(\mathbb{F}^{-1}(\mathbb{U})) \geq \alpha > 0$ and $\mathcal{T}_2(\mathbb{F}^{-1}(\mathbb{V})) \leq \beta < 1$. Hence $\mathbb{F}$ is $(0, 1)$-weakly intuitionistic smooth fuzzy continuous on $\mathbb{A}$. □

Counter example 3.26 *There exists an intuitionistic fuzzy proper function* $\mathbb{F}$ *which is* $(0, 1)$*-weakly intuitionistic smooth fuzzy continuous but not* (α, β)*-weakly smooth fuzzy continuous for some* $(\alpha, \beta) \in \zeta^*$.

Let $X=\{x,y\}$, $Y=\{a,b\}$.
If $\mathbb{A}_{[r,s]}^{[(0.7,0.3),(0.5,0.4)]}$, $\mathbb{B}_{[a,b]}^{[(0.8,0.2),(0.6,0.4)]}$, $\mathbb{U}_{1[x,y]}^{[(0.6,0.4),(0.5,0.4)]}$, $\mathbb{V}_{1[a,b]}^{[(0.6,0.4),(0.6,0.4)]}$, then $\mathbb{U}_1 \ll \mathbb{A} \in \zeta^X$, $\mathbb{V}_1 \ll \mathbb{B} \in \zeta^Y$. We define smooth fuzzy topologies $\mathcal{T}=(\mathcal{T}_1,\mathcal{T}_2)$ on $\mathscr{I}_{\mathbb{A}}$ and $\mathcal{T}^{'}=(\mathcal{T}_1^{'},\mathcal{T}_2^{'})$ on $\mathscr{I}_{\mathbb{B}}$ by

$$\mathcal{T}_1(\mathbb{U})=\begin{cases}1, & \mathbb{U}=(0,1) \text{ or } \mathbb{A},\\ 0.6, & \mathbb{U}=\mathbb{U}_1,\\ 0, & \text{otherwise}\end{cases}$$

$$\mathcal{T}_2(\mathbb{U})=\begin{cases}0, & \mathbb{U}=(0,1) \text{ or } \mathbb{A},\\ 0.2, & \mathbb{U}=\mathbb{U}_1,\\ 1, & \text{otherwise}\end{cases}$$

and

$$\mathcal{T}_1^{'}(\mathbb{V})=\begin{cases}1, & \mathbb{V}=(0,1) \text{ or } \mathbb{B},\\ 0.7, & \mathbb{V}=\mathbb{V}_1,\\ 0, & \text{otherwise .}\end{cases}$$

$$\mathcal{T}_2^{'}(\mathbb{V})=\begin{cases}0, & \mathbb{V}=(0,1) \text{ or } \mathbb{B},\\ 0.3, & \mathbb{V}=\mathbb{V}_1,\\ 1, & \text{otherwise .}\end{cases}$$

It is clear that $(\mathbb{A},\mathcal{T})$, $(\mathbb{B},,\mathcal{T}^{'})$ are two intuitionistic smooth fuzzy topological spaces. Let the proper function $\mathbb{F}:(\mathbb{A},\mathcal{T})\to(\mathbb{B},\mathcal{T}^{'})$ be defined by

$$\mathbb{F}(x,a)=\mathbb{A}(x),\mathbb{F}(x,b)=(0,1),\mathbb{F}(y,a)=(0,1),\mathbb{F}(y,b)=\mathbb{A}(y).$$

We note that $\mathbb{F}^{-1}(\mathbb{V}_1)_{[x,y]}^{[(0.6,0.4),(0.5,0.4)]}$ and $\mathbb{F}^{-1}(\mathbb{B})_{[x,y]}^{[(0.7,0.3),(0.5,0.4)]}$. Therefore $\mathbb{F}$ is $(0,1)$-weakly intuitionistic smooth fuzzy continuous on $\mathbb{A}$. But $\mathbb{F}$ is not (α,β)-weakly smooth fuzzy continuous for $\alpha=0.65$ and any $\beta\in I$ with $(\alpha,\beta)\in\zeta^*$. Because $\mathcal{T}_1^{'}(\mathbb{V}_1)=0.7>\alpha$ but $\mathcal{T}_1(\mathbb{F}^{-1}(\mathbb{V}_1))=\mathcal{T}_1(\mathbb{U}_1)=0.6\not>\alpha$.

Theorem 3.27 *Let $\mathbb{F}:\mathbb{A}\to\mathbb{B}$ and $\mathbb{G}:\mathbb{B}\to\mathbb{C}$. Then Prove the following,*

1. *If $\mathbb{F}$ and $\mathbb{G}$ are intuitionistic smooth fuzzy continuous, then $\mathbb{G}\circ\mathbb{F}$ is intuitionistic smooth fuzzy continuous.*

2. *If $\mathbb{F}$ and $\mathbb{G}$ are intuitionistic weakly smooth fuzzy continuous, then $\mathbb{G}\circ\mathbb{F}$ is intuitionistic weakly smooth fuzzy continuous.*

3. *If $\mathbb{F}$ and $\mathbb{G}$ are intuitionistic α-weakly smooth fuzzy continuous, then $\mathbb{G}\circ\mathbb{F}$ is intuitionistic α-weakly smooth fuzzy continuous.*

4. *If $\mathbb{F}$ and $\mathbb{G}$ are intuitionistic $(0,1)$-weakly smooth fuzzy continuous, then $\mathbb{G}\circ\mathbb{F}$ is intuitionistic $(0,1)$-weakly smooth fuzzy continuous.*

5. *If $\mathbb{F}$ and $\mathbb{G}$ are intuitionistic (α,β)-weakly smooth fuzzy continuous, then $\mathbb{G}\circ\mathbb{F}$ is intuitionistic (α,β)-weakly smooth fuzzy continuous.*

Proof. Let $\mathbb{W} \ll \mathbb{C}$. We claim that $\mathcal{T}_1((G \circ F)^{-1}(\mathbb{W})) \gg \mathcal{T}_3(\mathbb{W})$.

$$\begin{aligned}\mathcal{T}_1((\mathbb{G} \circ \mathbb{F})^{-1}(\mathbb{W})) &= \mathcal{T}_1(\mathbb{F}^{-1}(\mathbb{G}^{-1}(\mathbb{W}))) \\ &\gg \mathcal{T}_2(\mathbb{G}^{-1}(\mathbb{W})) \\ &\gg \mathcal{T}_3(\mathbb{W}).\end{aligned}$$

Hence $G \circ F$ is intuitionistic smooth fuzzy continuous on $\mathbb{A}$. Similarly, we can prove the remaining statements. □

4 Projection maps

Definition 4.1 *Let $\mathbb{A}_j$ be an intuitionistic fuzzy set on X_j, for every $j \in J$. For every $k \in J$, the k^{th} projection map $\mathbb{P}_k : \prod \mathbb{A}_j \to \mathbb{A}_k$ defined as an intuitionistic fuzzy proper function such that for every $[x_j] \in \prod X_j$ and $y_k \in X_k$,*

$$\mathbb{P}_k([x_j], y_k) = \begin{cases} \prod \mathbb{A}_j([x_j]), & y_k = x_k \\ 0, & y_k \neq x_k. \end{cases}$$

Definition 4.2 (Cf. [21]) *Let $\{(\mathbb{A}_j, \tau_j) : j \in J\}$ be family of intuitionistic smooth fuzzy topological spaces and $\mathbb{P}_k : \prod \mathbb{A}_j \to \mathbb{A}_k$ denote the k^{th} projection map. Let $\mathscr{S} = \{\mathbb{P}_k^{-1}(\mathbb{U}_k) : \mathcal{T}_k(U_k) \in \zeta^*, k \in J$ and $\mathscr{B}_{\mathscr{S}}$ be the collection of all finite intersection of members of $\mathscr{S}$. We define $\mathcal{T} : \mathscr{S} \to \zeta$ by $\mathcal{T}(\mathbb{P}_k^{-1}(\mathbb{U}_k)) = \mathcal{T}_k(\mathbb{U}_k)$. We also define $\mathcal{T}_{\mathscr{S}}$ as follows,*

$$\mathcal{T}_{\mathscr{S}}(\mathbb{U}) = \begin{cases} \sqcap \{\mathcal{T}(\mathbb{E}_1), \mathcal{T}(\mathbb{E}_2)\}, & \mathbb{U} = \mathbb{E}_1 \sqcap \mathbb{E}_2 \text{ where } \mathbb{E}_1, \mathbb{E}_2 \in \mathscr{S}, \\ \sqcup \mathcal{T}(\mathbb{W}_i), & \mathbb{U} = \bigsqcup_i \mathbb{W}_i \text{ where each } \mathbb{W}_i \in \mathscr{B}_{\mathscr{S}} \\ 0, & \text{otherwise} \end{cases}$$

$\mathcal{T}_{\mathscr{S}}$ is called the product of $\mathcal{T}_j$'s and $(\prod X_j, \mathcal{T}_{\mathscr{S}})$ is called the product of intuitionistic smooth fuzzy topological spaces $\{(X_j, \mathcal{T}_j) : j \in J\}$.

Definition 4.3 *Let $\mathbb{A} \in \zeta^X$, $\mathbb{B}_j \in \zeta^{Y_j}$, $\forall j \in J$. If $\mathbb{F}_j : \mathbb{A} \to \mathbb{B}_j$, then we define $[\mathbb{F}_j] : \mathbb{A} \to \prod \mathbb{B}_j$ by*

$$([\mathbb{F}_j])(x, [y_j]) = \bigsqcap_{j \in J} \mathbb{F}_j(x, y_j), \ \forall (x, [y_j]) \in X \times \prod Y_j.$$

Lemma 4.4 *Let $\mathbb{P}_k : \prod \mathbb{A}_j \to \mathbb{A}_k$ be projection map. If $\mathbb{U}_k \ll \mathbb{A}_k$, then $\mathbb{P}_k^{-1}(\mathbb{U}_k) = \prod_{j \neq k} \mathbb{A}_j \times \mathbb{U}_k$.*

Proof. For an arbitrary $[x_j] \in \prod X_j$,

$$\begin{aligned}
\mathbb{P}_k^{-1}(\mathbb{U}_k)([x_j])) &= \prod \mathbb{A}_j([x_j]) \sqcap \mathbb{U}_k(x_k) \\
&= \underset{j\in J}{\sqcap} \mathbb{A}_j(x_j)) \sqcap \mathbb{U}_k(x_k) \\
&= (\bigwedge_{j\in J} \mu_{\mathbb{A}_j}(x_j), \bigvee_{j\in J} \nu_{\mathbb{A}_j}(x_j)) \sqcap (\mu_{\mathbb{U}_k}(x_k) \wedge \nu_{\mathbb{U}_k}(x_k)) \\
&= (\bigwedge_{j\in J} \mu_{\mathbb{A}_j}(x_j) \wedge \mu_{\mathbb{U}_k}(x_k), \bigvee_{j\in J} \nu_{\mathbb{A}_j}(x_j) \vee \nu_{\mathbb{U}_k}(x_k)) \\
&= (\bigwedge_{j\neq k} \mu_{\mathbb{A}_j}(x_j) \wedge \mu_{\mathbb{U}_k}(x_k), \bigvee_{j\neq k} \nu_{\mathbb{A}_j}(x_j) \vee \nu_{\mathbb{U}_k}(x_k)) \\
&= (\bigwedge_{j\neq k} \mu_{\mathbb{A}_j}(x_j), \bigvee_{j\neq k} \nu_{\mathbb{A}_j}(x_j)) \sqcap (\mu_{\mathbb{U}_k}(x_k) \wedge \nu_{\mathbb{U}_k}(x_k)) \\
&= (\prod_{j\neq k} \mathbb{A}_j \times \mathbb{U}_k)([x_j]).
\end{aligned}$$

Hence the lemma. □

Theorem 4.5 *Let $\mathbb{P}_k : \prod \mathbb{A}_j \to \mathbb{A}_k$ be the kth projection, where $k \in J$. Then prove the following.*

1. *$\mathbb{P}_k$ is intuitionistic smooth fuzzy continuous on $\prod \mathbb{A}_j$.*
2. *$\mathbb{P}_k$ is intuitionistic weakly smooth fuzzy continuous on $\prod \mathbb{A}_j$.*
3. *$\mathbb{P}_k$ is intuitionistic α-weakly smooth fuzzy continuous on $\prod \mathbb{A}_j$, $\forall \alpha \in (0, 1]$.*
4. *$\mathbb{P}_k$ is intuitionistic $(0, 1)$weakly smooth fuzzy continuous on $\prod \mathbb{A}_j$.*
5. *$\mathbb{P}_k$ is intuitionistic (α, β)-weakly smooth fuzzy continuous on $\prod \mathbb{A}_j$, $\forall(\alpha, \beta) \in \zeta^*$.*

Proof. Let $\mathbb{U}_k \ll \mathbb{A}_k$. From the previous lemma, and by using the definition of product topology $\mathcal{T}$ on intuitionistic smooth fuzzy topological space, we have $\mathcal{T}(\mathbb{P}_k^{-1}(\mathbb{U}_k)) = \mathcal{T}(\prod_{j\neq k} \mathbb{A}_j \times \mathbb{U}_k) = \mathcal{T}_k(\mathbb{U}_k)$. From this fact the theorem immediately follows. □

Lemma 4.6 *If $\mathbb{F} : \mathbb{A} \to \prod \mathbb{B}_j$, then $\mathbb{F} = [\mathbb{P}_j \circ \mathbb{F}]$, where $\mathbb{P}_k : \prod \mathbb{B}_j \to \mathbb{B}_k$ be the kth projection map for every $k \in J$.*

Proof. Let $x \in X$ be arbitrary. By definition, there exists unique $[y_j] \in \prod Y_j$ is such that $\mathbb{F}(x, [y_j]) = \mathbb{A}(x)$. To prove this lemma, we shall show that

$$([\mathbb{P}_j \circ \mathbb{F}])(x, [z_j]) = \begin{cases} \mathbb{A}(x) & [z_j] = [y_j] \\ (0, 1) & [z_j] \neq [y_j]. \end{cases}$$

Assume that $[z_j]=[y_j]$. Let $k\in J$ be arbitrary. Then we have $z_k=y_k$ and hence

$$\begin{aligned}(\mathbb{P}_k\circ\mathbb{F})(x,z_k) &= (\mathbb{P}_k\circ\mathbb{F})(x,y_k)\\ &= \bigsqcup_{[w_j]\in\prod Y_j}\{\mathbb{F}(x,[w_j])\sqcap\mathbb{P}_k([w_j],y_k)\}\\ &= \mathbb{F}(x,[y_j])\sqcap\mathbb{P}_k([y_j],y_k)\\ &= \mathbb{A}(x)\sqcap\prod\mathbb{B}_j([y_j])\\ &= \mathbb{A}(x).\ (\text{ since } \mathbb{F}(x,[y_j])=\mathbb{A}(x)\ll\prod\mathbb{B}_j([y_j]))\end{aligned}$$

Therefore $([\mathbb{P}_j\circ\mathbb{F}])(x,[z_j])=\sqcap_{j\in J}(\mathbb{P}_j\circ\mathbb{F})(x,z_j)=\sqcap_{j\in J}\mathbb{A}(x)=\mathbb{A}(x)$.

Next assume that $[z_j]\neq[y_j]$. Then there exists $i\in J$ such that $z_i\neq y_i$. Now

$$\begin{aligned}(\mathbb{P}_i\circ\mathbb{F})(x,z_i) &= \bigsqcup_{[w_j]\in\prod Y_j}\{\mathbb{F}(x,w_j))\sqcap\mathbb{P}_j([w_j],z_j)\}\\ &= \mathbb{F}(x,[y_j])\sqcap\mathbb{P}_i([y_j],z_i)\\ &= (0,1).\end{aligned}$$

Thus $([\mathbb{P}_j\circ\mathbb{F}])(x,[z_j])=\sqcap_{j\in J}(\mathbb{P}_j\circ\mathbb{F})(x,z_j)=(0,1)$, since $i\in J$. □

Theorem 4.7 *Let* $\mathbb{F}:(\mathbb{A},\sigma)\to(\prod\mathbb{A}_j,\mathcal{T})$ *be an intuitionistic fuzzy proper function such that* $\mathbb{F}=[\mathbb{F}_j]$*, where* $\mathbb{F}_j:(\mathbb{A},\sigma)\to(\mathbb{A}_j,\mathcal{T}_j)$*, for every* $j\in J$*. If* $\mathbb{F}$ *is intuitionistic smooth fuzzy continuous on* $\mathbb{A}$*, then* $\mathbb{F}_j$ *is intuitionistic smooth fuzzy continuous for every* $j\in J$ *on* $\mathbb{A}$*.*

Proof. Assume that $\mathbb{F}$ is intuitionistic smooth fuzzy continuous. By using Lemma 4.6, we have $\mathbb{F}_j=\mathbb{P}_j\circ\mathbb{F}$, for every $j\in J$. Since each $\mathbb{P}_j$ is intuitionistic smooth fuzzy continuous and $\mathbb{F}$ is intuitionistic smooth fuzzy continuous, we get $\mathbb{F}_j=\mathbb{P}_j\circ\mathbb{F}$ is intuitionistic smooth fuzzy continuous on $\mathbb{A}$, for every $j\in J$.
The converse of the above theorem is not true.

Counter example 4.8 *Let* $X=\{x,y\}$*,* $Y=\{a,b\}$*. Define* $\mathbb{A}\in\zeta^X$ *by* $\mathbb{A}^{[(0.6,0.3),(0.7,0.2)]}_{[x,y]}$ *and* $\mathbb{B}_1,\mathbb{B}_2\in\zeta^Y$ *by* $\mathbb{B}_1{}^{[(0.8,0.2),(0.7,0.3)]}_{[a,b]}$ *and* $\mathbb{B}_2{}^{[(0.7,0.2),(0.9,0.1)]}_{[a,b]}$*. Define the intuitionistic fuzzy subsets* $\mathbb{U}_n\ll\mathbb{A}$*,* $n=1,2,3\ldots$*,* $\mathbb{V}_0\ll\mathbb{B}_1$ *and* $\mathbb{W}_n\ll\mathbb{B}_2$*,* $n=1,2,3\ldots$ *by* $\mathbb{U}_n{}^{[(0.6-\frac{1}{n+9},0.4-\frac{1}{n+9}),(0.5-\frac{1}{n+9},0.4-\frac{1}{n+9}]}_{[x,y]}$, $\mathbb{V}_0{}^{[(0.5,0.3),(0.4,0.3)]}_{[a,b]}$, $\mathbb{W}_n{}^{[(0.5-\frac{1}{n+9},0.4-\frac{1}{n+9}),(0.6-\frac{1}{n+9},0.4-\frac{1}{n+9})]}_{[a,b]}$.

We define $\sigma=(\sigma_1,\sigma_2):\mathscr{I}_{\mathbb{A}}\to\zeta$ *by*

$$\sigma_1(\mathbb{U})=\begin{cases}1, & \mathbb{U}=(0,1)\ \textit{or}\ \mathbb{A},\\ \frac{1}{n+1}, & \mathbb{U}=\mathbb{U}_n,\ n=1,2,3,\ldots\\ 0, & \textit{otherwise}\ ,\end{cases}$$

$$\sigma_2(\mathbb{U})=\begin{cases}0, & \mathbb{U}=(0,1)\ \textit{or}\ \mathbb{A},\\ \frac{1}{n+1}, & \mathbb{U}=\mathbb{U}_n,\ n=1,2,3,\ldots\\ 1, & \textit{otherwise}\ ,\end{cases}$$

$\mathcal{T}^{'} = (\mathcal{T}_1^{'}, \mathcal{T}_2^{'}) : \mathscr{I}_{\mathbb{B}_1} \to \zeta$ *by*

$$\mathcal{T}_1^{'}(\mathbb{V}) = \begin{cases} 1, & \mathbb{V} = (0,1) \ or \ \mathbb{B}_1, \\ 0.1, & \mathbb{V} = \mathbb{V}_0, \\ 0, & otherwise \ , \end{cases}$$

$$\mathcal{T}_2^{'}(\mathbb{V}) = \begin{cases} 0, & \mathbb{V} = (0,1) \ or \ \mathbb{B}_1, \\ 0.6, & \mathbb{V} = \mathbb{V}_0, \\ 1, & otherwise \ , \end{cases}$$

$\mathcal{T}^{"} = (\mathcal{T}_1^{"}, \mathcal{T}_2^{"}) : \mathscr{I}_{\mathbb{B}_2} \to \zeta$ *by*

$$\mathcal{T}_1^{"}(\mathbb{W}) = \begin{cases} 1, & \mathbb{W} = (0,1) \ or \ \mathbb{B}_2, \\ \frac{1}{n+1}, & \mathbb{W} = \mathbb{W}_n, \ n = 1,2,3,\ldots \\ 0, & otherwise \ , \end{cases}$$

$$\mathcal{T}_2^{"}(\mathbb{W}) = \begin{cases} 0, & \mathbb{W} = (0,1) \ or \ \mathbb{B}_2, \\ \frac{1}{n+1}, & \mathbb{W} = \mathbb{W}_n, \ n = 1,2,3,\ldots \\ 1, & otherwise \ . \end{cases}$$

then $(\mathbb{B}_1, \mathcal{T}^{'})$ *and* $(\mathbb{B}_2, \mathcal{T}^{"})$ *are intuitionistic smooth fuzzy topological spaces. We denote the product of the ISFTSs* $(\mathbb{B}_1, \mathcal{T}^{'})$ *and* $(\mathbb{B}_2, \mathcal{T}^{"})$ *by* $\mathcal{T}$. *Define an intuitionistic fuzzy proper function* $\mathbb{F} : (\mathbb{A}, \sigma) \to (\mathbb{B}_1 \times \mathbb{B}_2, \mathcal{T})$ *by*

$$\mathbb{F}(x,(a,b)) = \mathbb{A}(x), \mathbb{F}(x,(a,a)) = (0,1), \mathbb{F}(x,(b,a)) = (0,1), \mathbb{F}(x,(b,b)) = (0,1);$$

and

$$\mathbb{F}(y,(b,a)) = \mathbb{A}(y), \mathbb{F}(y,(a,a)) = (0,1), \mathbb{F}(x,(a,b)) = (0,1), \mathbb{F}(x,(b,b)) = (0,1);$$

Let $\mathbb{U} = \bigsqcup_{n \in \mathbb{N}} (\mathbb{B}_1 \times \mathbb{W}_n)$. *Since* $\sigma\left(\mathbb{F}^{-1}(\mathbb{U})^{[(0.6,0.3),(0.5,0.3)]}_{[x,y]}\right) = (0,1)$, $\mathbb{T}_{\mathscr{S}}(\mathbb{U}) = (0.5, 0)$, *we get* $\mathbb{F}$ *is not intuitionistic smooth fuzzy continuous on* $\mathbb{A}$. *But* $\mathbb{F}_i : (\mathbb{A}, \sigma) \to (\mathbb{B}_i, \mathcal{T}_i)$ *are intuitionistic smooth fuzzy continuous for* $i = 1, 2$.

We prove the following theorems like Theorem 4.7

Theorem 4.9 *Let* $\mathbb{F} : (\mathbb{A}, \sigma) \to (\prod \mathbb{A}_j, \mathcal{T})$ *be proper function such that* $\mathbb{F} = [\mathbb{F}_j]$, *where* $\mathbb{F}_j : (\mathbb{A}, \sigma) \to (\mathbb{A}_j, \mathcal{T}_j)$, *for every* $j \in J$. *If* $\mathbb{F}$ *is intuitionistic weakly smooth fuzzy continuous on* $\mathbb{A}$, *then* $\mathbb{F}_j$ *is intuitionistic weakly smooth fuzzy continuous for every* $j \in J$ *on* $\mathbb{A}$.

Theorem 4.10 *Let* $\mathbb{F} : (\mathbb{A}, \sigma) \to (\prod \mathbb{A}_j, \mathcal{T})$ *be proper function such that* $\mathbb{F} = [\mathbb{F}_j]$, *where* $\mathbb{F}_j : (\mathbb{A}, \sigma) \to (\mathbb{A}_j, \mathcal{T}_j)$, *for every* $j \in J$. *If* $\mathbb{F}$ *is intuitionistic (0,1)-weakly smooth fuzzy continuous on* $\mathbb{A}$, *then* $\mathbb{F}_j$ *is intuitionistic (0,1)-weakly smooth fuzzy continuous for every* $j \in J$ *on* $\mathbb{A}$.

Counter example 4.11 *The converse of the above theorems are not true.*

See the same function given in Counter example 4.8.
The converse holds when we use positive minimum intuitionistic smooth fuzzy topology in the domain of $\mathbb{F}$ or when $\mathbb{F}$ is intuitionistic α-weakly smooth fuzzy continuous or $\mathbb{F}$ is intuitionistic (α,β)-weakly smooth fuzzy continuous.

Theorem 4.12 *Let $(\mathbb{A},\sigma)$ be a positive minimum intuitionistic smooth fuzzy topological space and $(\mathbb{A}_j,\mathcal{T}_j)$ be intuitionistic smooth fuzzy topological spaces. A fuzzy proper function $\mathbb{F}:(\mathbb{A},\sigma)\to(\prod\mathbb{A}_j,\mathcal{T})$ is intuitionistic $(0,1)$-weakly smooth fuzzy continuous if and only if $\mathbb{F}_j$ is intuitionistic $(0,1)$-weakly smooth fuzzy continuous for every $j\in J$.*

Proof. We prove only that if $\mathbb{F}_j$ is intuitionistic $(0,1)-$weakly smooth fuzzy continuous on $\mathbb{A}$, for every $j\in J$, then $\mathbb{F}$ is intuitionistic $(0,1)$-weakly smooth fuzzy continuous on $\mathbb{A}$. Let $\mathbb{B}\in\mathscr{B}_{\mathscr{S}}$. Then $\mathbb{B}=\prod\mathbb{U}_j$, where $\mathbb{U}_j=\mathbb{A}_j$, $\forall j\in J$ with $j\neq j_1,j_2,\ldots j_n$ and $\mathcal{T}_j(U_j)\gg(0,1)$, $\forall j=j_1,j_2,\ldots j_n.$, for some $j_1,j_2,\ldots j_n\in J$. We claim that $\mathbb{F}^{-1}(\mathbb{B})=\underset{j\in J}{\sqcap}(\mathbb{P}_j\circ\mathbb{F})^{-1}(\mathbb{U}_j)$.

$$\begin{aligned}
(\mathbb{F}^{-1}(\mathbb{B}))(x) &= \mathbb{A}(x)\sqcap\mathbb{B}([x_j]) \\
&\quad \text{where } [x_j]\in\prod X_j \text{ is such that } \mathbb{F}(x,[x_j])=\mathbb{A}(x) \\
&= \mathbb{A}(x)\sqcap\prod\mathbb{U}_j([x_j])=\mathbb{A}(x)\sqcap\underset{j\in J}{\sqcap}\mathbb{U}_j(x_j) \\
&= \underset{j\in J}{\sqcap}(\mathbb{A}(x)\sqcap\mathbb{U}_j(x_j))=\underset{j\in J}{\sqcap}((\mathbb{P}_j\circ\mathbb{F})^{-1}(\mathbb{U}_j))(x) \\
&= (\underset{j\in J}{\sqcup}\mathbb{F}_j^{-1}(\mathbb{U}_j))(x) \quad (\text{ since } \mathbb{P}_j\circ\mathbb{F}=\mathbb{F}_j\ )
\end{aligned}$$

Therefore

$$\begin{aligned}
\sigma(\mathbb{F}^{-1}(\mathbb{B})) &= \sigma\left(\underset{j\in J}{\sqcup}\mathbb{F}_j^{-1}(\mathbb{U}_j)\right) \\
&= \sigma\left(\overset{n}{\underset{i=1}{\sqcup}}\mathbb{F}_{j_i}^{-1}(\mathbb{U}_{j_i})\sqcap\underset{j\neq j_1,j_2,\ldots j_n}{\sqcap}\mathbb{F}_j^{-1}(\mathbb{A}_j)\right) \\
&= \sigma\left(\overset{n}{\underset{i=1}{\sqcap}}\mathbb{F}_{j_i}^{-1}(\mathbb{U}_{j_i})\right) \quad (\text{ since } \mathbb{F}_j^{-1}(\mathbb{A}_j)=\mathbb{A},\ \forall j\ ) \\
&\gg \overset{n}{\underset{i=1}{\sqcap}}\sigma(\mathbb{F}_{j_i}^{-1}(\mathbb{U}_{j_i})).
\end{aligned}$$

Since $\mathbb{F}_{j_i}$ is intuitionistic $(0,1)$-weakly smooth fuzzy continuous, we have $\sigma(\mathbb{F}_{j_i}^{-1}(\mathbb{U}_{j_i})\gg(0,1)$, for every $i=1,2,\ldots n$. Hence $\sigma(\mathbb{F}^{-1}(\mathbb{B}))\gg(0,1)$.

If $\mathbb{U}\in\prod\mathbb{A}_j$ with $\mathcal{T}(\mathbb{U})\gg(0,1)$, then $\mathbb{U}=\underset{k\in K}{\sqcup}\mathbb{B}_k$, where each $\mathbb{B}_k\in\mathscr{B}_{\mathscr{S}}$ and K is any index set. Since $\sigma(\mathbb{F}^{-1}(\mathbb{B}_k))\gg(0,1),\forall k\in K$, and σ is a positive minimum intuitionistic smooth fuzzy topology, we get $\sigma(\mathbb{F}^{-1}(\mathbb{U}))=\sigma(\mathbb{F}^{-1}(\underset{k\in K}{\sqcup}\mathbb{B}_k))=\sigma(\underset{k\in K}{\sqcup}\mathbb{F}^{-1}(\mathbb{B}_k))\gg\underset{k\in K}{\sqcap}\sigma(\mathbb{F}^{-1}(\mathbb{B}_k))\gg(0,1)$. □

Similarly we can prove the following theorems.

Theorem 4.13 *Let $\alpha\in(0,1]$. A intuitionistic fuzzy proper function $\mathbb{F}:\mathbb{A}\to\prod\mathbb{B}_j$ is α-weakly intuitionistic smooth fuzzy continuous if and only if $\mathbb{F}_j$ is α-weakly intuitionistic smooth fuzzy continuous for every $j\in J$.*

Theorem 4.14 *Let $(\alpha,\beta)\in\zeta^*$. An intuitionistic fuzzy proper function $\mathbb{F}:\mathbb{A}\to\prod\mathbb{B}_j$ is (α,β)-weakly intuitionistic smooth fuzzy continuous if and only if $\mathbb{F}_j$ is (α,β)-weakly intuitionistic smooth fuzzy continuous for every $j\in J$.*

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