19th Int. Workshop on IFSs, Burgas, 4–6 June 2015
Notes on Intuitionistic Fuzzy Sets
ISSN 1310–4926
Vol. 21, 2015, No. 2, 19–24

Some properties of the matrix representation of the intuitionistic fuzzy modal operators

Sinem Yilmaz[1], Mehmet Çitil and Gökhan Çuvalcioğlu[1]

[1] Department of Mathematics, University of Mersin
Mersin, Turkey
e-mails: sinemnyilmaz@gmail.com,
gcuvalcioglu@gmail.com

[2] University of Sütçü İmam, Department of Mathematics
K. Maraş, Turkey
e-mail: citil@ksu.edu.tr

Abstract: In 1965, Fuzzy Set Theory was defined by L. Zadeh as an extension of crisp sets [2]. K. T. Atanassov generalized fuzzy sets in to Intuitionistic Fuzzy Sets in 1983[2]. Intuitionistic Fuzzy Modal Operator was firstly defined in [2] and the other operators were defined by different autors. Some properties of them have studied until now. Matrix representation of Intuitionistic Fuzzy Modal Operators were defined and some algebraic properties were given in [2]. In this study, we examined some properties of these matrix representations and we obtained some relationships between them.
Keywords: Intuitionistic fuzzy sets, Intuitionistic fuzzy modal operators, Matrix representation of IFS operators.
AMS Classification: 03E72, 47S40.

1 Introduction

Let X be a fixed set. Function $\mu : X \to [0, 1]$ is called a fuzzy set over X [7]. The class of the fuzzy sets over X is denoted by $FS(X)$. For $x \in X$, $\mu(x)$ is the membership degree of x and the non-membership degree is $1 - \mu(x)$. Intuitionistic Fuzzy Sets (IFSs) have been introduced in [1], as an extension of fuzzy sets.

Definition 1. *[2] Let L=[0,1] then*

$$L^* = \{(x_1, x_2) \in [0,1]^2 : x_1 + x_2 \leq 1\}$$

is a lattice with $(x_1, x_2) \leq (y_1, y_2) :\Longleftrightarrow$ "$x_1 \leq y_1$ *and* $x_2 \geq y_2$".
For $(x_1, y_1), (x_2, y_2) \in L^*$,the operators $\wedge$ and $\vee$ on $(L^*, \leq)$ are defined as following;

$$(x_1, y_1) \wedge (x_2, y_2) = (\min(x_1, x_2), \max(y_1, y_2))$$

$$(x_1, y_1) \vee (x_2, y_2) = (\max(x_1, x_2), \min(y_1, y_2))$$

Definition 2. *[1] An intuitionistic fuzzy set (shortly IFS) on a set X is an object of the form*

$$A = \{\langle x, \mu_A(x), \nu_A(x)\rangle : x \in X\}$$

where $\mu_A(x), (\mu_A : X \to [0,1])$ is called the "degree of membership of x in A ", $\nu_A(x), (\nu_A : X \to [0,1])$is called the " degree of non- membership of x in A ",and where μ_A and ν_A satisfy the following condition:

$$\mu_A(x) + \nu_A(x) \leq 1, \text{ for all } x \in X.$$

The hesitation, indeterminacy, or uncertainty degree of x is defined by $\pi_A(x) = 1 - \mu_A(x) - \nu_A(x)$

Definition 3. *[1]An IFS A is said to be contained in an IFS B (notation $A \sqsubseteq B$) if and only if for all $x \in X : \mu_A(x) \leq \mu_B(x)$ and $\nu_A(x) \geq \nu_B(x)$.*

It is clear that $A = B$ if and only if $A \sqsubseteq B$ and $B \sqsubseteq A$.

Definition 4. *[1]Let $A \in IFS$ and let $A = \{\langle x, \mu_A(x), \nu_A(x)\rangle : x \in X\}$ then the above set is called the complement of A*

$$A^c = \{\langle x, \nu_A(x), \mu_A(x)\rangle : x \in X\}$$

2 The matrix representation of intuitionistic fuzzy modal operators

The notion of Intuitionistic Fuzzy Operators (IFO) was intoduced first in [2]. After that several autors defined new Intuitionistic Fuzzy Operators and some properties of these operators were examined [6, 3, 4]. Intuitionistic Fuzzy Operators were studied with matrices in [5] and some algebraic properties of them were given. In this study, we examined some properties of matrix representations and we obtained some relationships between them.

Notation 1. $(\max)(\min)\{a,b\}$ *has property* P *if and only if* $(\max\{a,b\}$ *has property* P *and* $\wedge(\min\{a,b\}$ *has property* P.

Let for brevity (a_{ij}) denote a matrice with elements, denoted also by a and let $M_{3\times 3}(\mathbb{R})$ be the set of (3 × 3)-matrices with elements – real numbers.

Let X be a fixed set. Then Ω and Γ are defining as following;

$$\Omega = \{\Theta \mid \Theta : IFS(X) \to IFS(X) \text{ is an IFMO}\}$$

$$\Gamma = \{(a_{ij}) : (a_{ij}) \in M_{3\times 3}(\mathbb{R}) \;\&\; 0 \le (\max)(\min)\{a_{11}+a_{12}, a_{21}+a_{22}\} \le 1$$
$$\& \; 0 \le a_{31}+a_{32} \le 1\}.$$

Definition 5. *[5]Let* X *be a set and* $A = \{\langle x, \mu_A(x), \nu_A(x)\rangle : x \in X\} \in IFS(X)$.

The mapping $\varphi_A : \Gamma \to \Omega$,

$$\varphi_A((a_{i,j})) = \{\langle x, a_{11}\mu_A(x) + a_{21}\nu_A(x) + a_{31}, a_{12}\mu_A(x) + a_{22}\nu_A(x) + a_{32}\rangle :$$
$$x \in X \;\&\; 0 \le (\max)(\min)\{a_{11}+a_{12}, a_{21}+a_{22}\} + a_{31}+a_{32} \le 1 \;\&\; 0 \le a_{31}+a_{32} \le 1\}.$$

After this we show the IFMOs with matrices as follows. Let $a_{11}, a_{21}, a_{31}, a_{12}, a_{22}, a_{32} \in [-1, 1]$ satisfy inequalities

$$0 \le (\max)(\min)\{a_{11}+a_{12}, a_{21}+a_{22}\} + a_{31}+a_{32} \le 1$$

and

$$0 \le a_{31}+a_{32} \le 1.$$

The coefficients $a_{11}, a_{21}, a_{31}, a_{12}, a_{22}, a_{32}$ may be negative so we studied with interval $[-1, 1]$ instead of $[0, 1]$.

Then

$$\Theta(A) = \{\langle x, a_{11}\,\mu_A\,(x) + a_{21}\,\nu_A(x) + a_{31},\; a_{12}\,\mu_A\,(x) + a_{22}\,\nu_A\,(x) + a_{32}\rangle : x \in X\}$$
$$= \begin{bmatrix} \mu_A\,(x) & \nu_A\,(x) & 1 \end{bmatrix} \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \\ a_{31} & a_{32} \end{bmatrix}$$

It is clear that for the present case, sets

$$\Theta = \{(a_{ij}) : (a_{ij}) \in M_{3\times 2}(\mathbb{R}) \;\&\; (a_{ij}) = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \\ a_{31} & a_{32} \end{bmatrix}\}$$

and

$$\Theta = \{(a_{ij}) : (a_{ij}) \in M_{3\times 3}(\mathbb{R}) \;\&\; (a_{ij}) = \begin{bmatrix} a_{11} & a_{12} & 0 \\ a_{21} & a_{22} & 0 \\ a_{31} & a_{32} & 1 \end{bmatrix}\}$$

are equal.

For brevity, in this paper, if $\varphi_A((a_{ij})) = \Theta$, we will use the notation $\Theta = \begin{bmatrix} a_{11} & a_{12} & 0 \\ a_{21} & a_{22} & 0 \\ a_{31} & a_{32} & 1 \end{bmatrix}$.

3 Some properties of the matrix representations

Thanks to some operations defined on matrices, various features of the the operators can be examined. Therefore, elementary row operations can be applied to the augmented matrix of matrix representations of Intuitionistic Fuzzy Modal Operators and we can determine their properties. Firstly we can give the following proposition.

Proposition 1. *Let X be a set, $A \in IFS(X)$ and $\Theta, \partial \in \Omega$.If*

$$\Theta(A) = \{\langle x, a_{11}\, \mu_A\,(x) + a_{31},\ a_{22}\, \nu_A\,(x) + a_{32}\rangle : x \in X\}$$

and

$$\partial(A) = \{\langle x, b_{11}\, \mu_A\,(x) + b_{31},\ b_{22}\, \nu_A\,(x) + b_{32}\rangle : x \in X\}$$

$$\left[I\colon \Theta^k \partial^n\right] \backsim \left[\Theta\colon \Theta^{k+1}\partial^n\right]$$

for all $k, n \in \mathbb{N}$.

Proof. $\left[I\colon \Theta^k \partial^n\right] =$

$$= \left[\begin{array}{ccc} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{array} \vdots \begin{array}{ccc} (a_{11})^k (b_{11})^n & 0 & 0 \\ 0 & (a_{22})^k\ (b_{22})^n & 0 \\ \left((b_{11})^n a_{3,1} \sum_{i=0}^{k-1} (a_{11})^i\right) + (b_{31} \sum_{i=0}^{n-1} (b_{11})^i) & ((b_{22})^n a_{32} \sum_{i=0}^{k-1} (a_{22})^i) + (b_{32} \sum_{i=0}^{n-1} (b_{22})^i) & 1 \end{array}\right]$$

If we make the appropriate elementary row operations then,

$$\left[\begin{array}{ccc} a_{11} & 0 & 0 \\ 0 & a_{22} & 0 \\ a_{31} & a_{32} & 1 \end{array} \vdots \begin{array}{ccc} (a_{11})^{k+1} (b_{11})^n & 0 & 0 \\ 0 & (a_{22})^{k+1}\ (b_{22})^n & 0 \\ (b_{11})^n a_{31}\big(\ (a_{11})^k + \sum_{i=0}^{k-1} (a_{11})^i\big) + b_{31} \sum_{i=0}^{n-1} (b_{11})^i & (b_{22})^n a_{32}\big(\ (a_{22})^k + \sum_{i=0}^{k-1} (a_{22})^i\big) + b_{32} \sum_{i=0}^{n-1} (b_{22})^i & 1 \end{array}\right]$$

$$\backsim \left[\begin{array}{ccc} a_{11} & 0 & 0 \\ 0 & a_{22} & 0 \\ a_{31} & a_{32} & 1 \end{array} \vdots \begin{array}{ccc} (a_{11})^{k+1} (b_{11})^n & 0 & 0 \\ 0 & (a_{22})^{k+1}\ (b_{22})^n & 0 \\ ((b_{11})^n a_{31} \sum_{i=0}^{k} (a_{11})^i) + b_{31} \sum_{i=0}^{n-1} (b_{11})^i & ((b_{22})^n a_{32} + \sum_{i=0}^{k} (a_{22})^i) + b_{32} \sum_{i=0}^{n-1} (b_{22})^i & 1 \end{array}\right]$$

$= \left[\Theta\colon \Theta^{k+1}\partial^n\right]$ □

Theorem 1. *Let X be a set, $A \in IFS(X)$ and $\Theta, \partial \in \Omega$.If*

$$\Theta(A) = \{\langle x, a_{11}\, \mu_A\,(x) + a_{31},\ a_{22}\, \nu_A\,(x) + a_{32}\rangle : x \in X\}$$

and

$$\partial(A) = \{\langle x, b_{11}\, \mu_A\,(x) + b_{31},\ b_{22}\, \nu_A\,(x) + b_{32}\rangle : x \in X\}$$

then

$$\left[\Theta\colon \Theta^{k+1}\partial^n\right] \backsim \left[\Theta^r\colon \Theta^{r+k}\partial^n\right]$$

for all $k, r, n \in \mathbb{N}$.

Proof. $\left[I\colon \Theta^k\partial^n\right] =$

$$\begin{bmatrix} 1 & 0 & 0 & \cdot & (a_{11})^k(b_{11})^n & 0 & 0 \\ 0 & 1 & 0 & \cdot & 0 & (a_{22})^k\,(b_{22})^n & 0 \\ 0 & 0 & 1 & \cdot & \left((b_{11})^n a_{3,1}\sum_{i=0}^{k-1}(a_{11})^i\right)+\left(b_{3,1}\sum_{i=0}^{n-1}(b_{11})^i\right) & ((b_{22})^n a_{32}\sum_{i=0}^{k-1}(a_{22})^i)+(b_{32}\sum_{i=0}^{n-1}(b_{22})^i) & 1 \end{bmatrix}$$

If we make the appropriate elementary row operations then,

$$\begin{bmatrix} (a_{11})^r & 0 & 0 & \vdots & (a_{11})^{k+r}(b_{11})^n & 0 & 0 \\ 0 & (a_{22})^r & 0 & \vdots & 0 & (a_{22})^{k+r}\,(b_{22})^n & 0 \\ a_{31}\sum_{i=0}^{r-1}(a_{11})^i & a_{32}\sum_{i=0}^{r-1}(a_{22})^i & 1 & \vdots & ((a_{11})^k a_{31}\sum_{i=0}^{r-1}(a_{11})^i)+ ((b_{11})^n a_{31}\sum_{i=0}^{k-1}(a_{11})^i)+(b_{31}\sum_{i=0}^{n-1}(b_{11})^i) & ((a_{22})^{k+r}(b_{22})^n a_{32}\sum_{i=0}^{r-1}(a_{22})^i)+ ((b_{22})^n a_{32}\sum_{i=0}^{k-1}(a_{22})^i)+(b_{32}\sum_{i=0}^{n-1}(b_{22})^i) & 1 \end{bmatrix}$$

$$\backsim \begin{bmatrix} (a_{11})^r & 0 & 0 & \vdots & (a_{11})^{k+r}(b_{11})^n & 0 & 0 \\ 0 & (a_{22})^r & 0 & \vdots & 0 & (a_{22})^{k+r}\,(b_{22})^n & 0 \\ a_{31}\sum_{i=0}^{r-1}(a_{11})^i & a_{32}\sum_{i=0}^{r-1}(a_{22})^i & 1 & \vdots & ((b_{11})^n a_{31}\sum_{i=0}^{k+r-1}(a_{11})^i) +(b_{31}\sum_{i=0}^{n-1}(b_{11})^i) & ((b_{22})^n a_{32}\sum_{i=0}^{k+r-1}(a_{22})^i) +(b_{32}\sum_{i=0}^{n-1}(b_{22})^i) & 1 \end{bmatrix}$$

$= \left[\Theta^r\colon \Theta^{r+k}\partial^n\right]$

and from proposition $\left[I\colon \Theta^k\partial^n\right] \backsim \left[\Theta\colon \Theta^{k+1}\partial^n\right]$.

So we obtain that $\left[\Theta\colon \Theta^{k+1}\partial^n\right] \backsim \left[\Theta^r\colon \Theta^{r+k}\partial^n\right]$. □

Corollary 1. *Let X be a set, $A \in IFS(X)$ and $\Theta \in \Omega$.*

If $\Theta(A) = \{\langle x, a_{11}\,\mu_A\,(x) + a_{31},\ a_{22}\,\nu_A\,(x) + a_{32}\rangle : x \in X\}$ then

$$\left[\Theta\colon \Theta^{k+1}\right] \backsim \left[\Theta^n\colon \Theta^{k+n}\right]$$

for all $k, n \in \mathbb{N}$.

Corollary 2. *Let X be a set, $A \in IFS(X)$ and $\Theta \in \Omega$.*

If $\Theta(A) = \{\langle x, a_{11}\,\mu_A\,(x) + a_{31},\ a_{22}\,\nu_A\,(x) + a_{32}\rangle : x \in X\}$ then

$$[\Theta\colon \Theta^n] = \left[\Theta^n\colon \Theta^{2n-1}\right]$$

for all $n \in \mathbb{N}^*$.

4 Open problems

1. Let $\mathcal{L} = \{\left[\Theta^k\colon \partial^n\right] : \Theta, \partial \in \Omega$ and $k, n \in \mathbb{N}\}$. Can we define any equivalence relation on set $\mathcal{L}$?
2. Can we define an algebraic structure which elements are $\left[\Theta^k\colon \partial^n\right]$ for $k, n \in \mathbb{N}$?

References

[1] Atanassov, K. (1983) Intuitionistic fuzzy sets, VII ITKR's Session, Sofia, June 1983 (Deposed in Central Sci. - Techn. Library of Bulg. Acad. of Sci., 1697/84) (in Bulgarian).

[2] Atanassov, K. T. (1999) *On Intuitionistic Fuzzy Sets Theory*, Springer, Heidelberg, 1999.

[3] Çuvalcıoğlu, G. (2013) On the Diagram of One Type Modal Operators on Intuitionistic Fuzzy Sets: Last Expanding with $Z^{\omega,\theta}_{\alpha,\beta}$, *Iranian J. of Fuzzy Systems*, 10(1), 89–106.

[4] Çuvalcıoğlu, G. (2013) The Extension of Modal Operators' Diagram with Last Operators, *Notes on Intuitionistic Fuzzy Sets*, 19(3), 56–61.

[5] Çuvalcıoğlu G., Yılmaz, S. & Atanassov, K. T. (2014) Matrix Representation of the Second Type of Intuitionistic Fuzzy Modal Operators, *Notes on Intuitionistic Fuzzy Sets*, 20(5), 9–16.

[6] Dencheva, K. (2004) Extension of intuitionistic fuzzy modal operators $\boxplus$ and $\boxtimes$, *Proc. of the Second Int. IEEE Symp. on Intelligent Systems*, Varna, 22–24 June 2004, Vol. 3, 21–22.

[7] Zadeh, L. A. (1965) Fuzzy sets, *Information and Control*, 8, 338–353.