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On Lagrange mean value theorem for functions on Atanassov IF-sets

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Abstract: On the family of IF sets [1] some elementary functions has been studied in [2, 3] as well as limit and continuity [4, 5]. In the present article we define derivation and with respect to the notion the Lagrange mean value theorem is formulated and proved.
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1 IF-sets

IF-sets have been introduced in [1] as a natural generalization of fuzzy sets with remarkable applications. Given a set Ω an IF set is a pair of functions (membership or non-membership resp.)

$$A = (\mu_A, \nu_A)$$

such that

$$\mu_A, \nu_A : \Omega \to [0, 1], \mu_A + \nu_A \leq 1.$$

Denote by $\mathcal{F}$ the family of all IF-sets. On $\mathcal{F}$ two binary operations $\oplus, \odot$ and one unary operation $\neg$ are defined:

$$A \oplus B = (\min(\mu_A + \mu_B, 1), \max(\nu_A + \nu_B - 1, 0)),$$

$$A \odot B = (\max(\mu_A + \mu_B - 1, 0), \min(\nu_A + \nu_B, 1)),$$

$$\neg A = (1 - \mu_A, 1 - \nu_A).$$

Further a partial ordering on $\mathcal{F}$ is given by

$$A \leq B \Longleftrightarrow \mu_A \leq \mu_B, \nu_A \geq \nu_B.$$

It is not difficult to construct an additive group $\mathcal{G} \supset \mathcal{F}$ with an ordering such that $\mathcal{G}$ is an lattice ordered group, where

$$A + B = (\mu_A + \mu_B, \nu_A + \nu_B - 1)$$

with the neutral element $0 = (0_\Omega, 1_\Omega)$ and

$$A \leq B \Longleftrightarrow \mu_A \leq \mu_B, \nu_A \geq \nu_B.$$

Lattice operations are given by

$$A \wedge B = (\mu_A \wedge \mu_B, \nu_A \vee \nu_B),$$

$$A \vee B = (\mu_A \vee \mu_B, \nu_A \wedge \nu_B),$$

Evidently

$$A - B = (\mu_A - \mu_B, \nu_A - \nu_B + 1).$$

The operations on $\mathcal{F}$ can be derived from operations on $\mathcal{G}$ if we use the unit $u = (1_\Omega, 0_\Omega)$:

$$A \oplus B = (A + B) \wedge u,$$

$$A \odot B = (A + B - u) \vee 0,$$

$$\neg A = u - A.$$

On our investigations also the following two operations will be used:

$$A.B = (\mu_A \mu_B, 1 - (1 - \nu_A)(1 - \nu_B)) =$$

$$= (\mu_A \mu_B, \nu_A + \nu_B - \nu_A \nu_B),$$

and if $\mu_B > 0, \nu_B < 1$, then

$$\frac{A}{B} = (\frac{\mu_A}{\mu_B}, 1 - \frac{1 - \nu_A}{1 - \nu_B}).$$

2 Differentiation

First we present a motivation. In [5] a real function f has been considered such that

$$[\mu_A, \mu_B] \cup [\nu_B, \nu_A] \subset Domf,$$

where $A \leq B$. Then the function $\bar{f} : [A, .B] \to R^2$ is defined by

$$\bar{f}(A) = (f(\mu_A), 1 - f(1 - \nu_A)).$$

Compute

$$X - X_0 = (\mu_X - \mu_{X_0}, \nu_X - \nu_{X_0} + 1)$$

$$\bar{f}(X) - \bar{f}(X_0) = (f(\mu_X), 1 - f(1 - \nu_X)) - (f(\mu_{X_0}), 1 - f(1 - \nu_{X_0})) =$$

$$= (f(\mu_X) - f(\mu_{X_0}), 1 - f(1 - \nu_X) - (1 - f(1 - \nu_{X_0})) + 1) =$$

$$= (f(\mu_X) - f(\mu_{X_0}), f(1 - \nu_{X_0}) - f(1 - \nu_X) + 1).$$

Therefore

$$\frac{\bar{f}(X) - \bar{f}(X_0)}{X - X_0} = (\frac{f(\mu_X) - f(\mu_{X_0})}{\mu_X - \mu_{X_0}}, 1 - \frac{1 - (f(1 - \nu_{X_0}) - f(1 - \nu_X) + 1)}{1 - (\nu_X - \nu_{X_0} + 1)} =$$

$$(\frac{f(\mu_X) - f(\mu_{X_0})}{\mu_X - \mu_{X_0}}, 1 - \frac{f(1 - \nu_X) - f(1 - \nu_{X_0})}{\nu_X - \nu_{X_0}}).$$

The above computation leads to the following definition.

Definition. Let $f'(x)$ exists whenever $x = \mu_A(u)$ or $x = 1 - \nu_A(v)$ for some $u, v \in \Omega$. Then we define

$$\bar{f}'(A) = (f'(\mu_A), 1 - f'(1 - \nu_A)).$$

Theorem. If $\bar{f}$ is differentiable in $X_0 \in \mathcal{G}$, then $\bar{f}$ is continuous in X_0.

Proof. By Theorem 1 of [5] $\bar{f}$ is continuous on [A,B] if and only if f is continuous on $[\mu_A, \mu_B]$ and $[\nu_B, \nu_A]$. Hence if $\bar{f}$ is differentiable, then f is differentiable. Therefore f is continuous, and $\bar{f}$ is continuous by [5].

3 Lagrange mean value theorem

Theorem. Let $\bar{f}$ be continuous on $[A, B]$, differentiable on (A, B). Then there exists $C \in (A, B)$ such that

$$\bar{f}(B) - \bar{f}(A) = \bar{f}'(C)(B - A).$$

Proof. By the definition

$$\bar{f}(B) - \bar{f}(A) = (f(\mu_B), 1 - f(1 - \nu_B)) - (f(\mu_A), 1 - f(1 - \nu_A)) =$$

$$= (f(\mu_B) - f(\mu_A), f(1 - \nu_A) - f(1 - \nu_B) + 1).$$

Fix $\omega \in \Omega$ and take $b \in \mu_B(\omega), a \in \mu_A(\omega)$. Then there exists $c \in (a, b)$ such that

$$f(b) - f(a) = f'(c)(b - a).$$

Define $\mu_C : \Omega \to R$ by the equality

$$\mu_C(\omega) = c,$$

hence we obtain $\mu_A \leq \mu_C \leq \mu_B$, and

$$f(\mu_B) - f(\mu_A) = f'(\mu_C)(\mu_B - \mu_A).$$

Similarly $\nu_C : \Omega \to R$ can be defined such that $1 - \nu_B \leq 1 - \nu_C \leq 1 - \nu_A$, and

$$f(1-\nu_A) - f(1-\nu_B) = f'(1-\nu_C)(1-\nu_A-(1-\nu_B)) =$$

$$= f'(1-\nu_C)(\nu_B-\nu_A).$$

Define $C = (\mu_C, \nu_C)$. Then $\mu_A \leq \mu_C \leq \mu_B, \nu_A \geq \nu_C \geq \nu_B$, hence $A \leq C \leq B$. Moreover

$$\bar{f}'(C) = (f'(\mu_C, 1 - f'(1-\nu_C)).$$

Therefore

$$\bar{f}(B) - \bar{f}(A) = (f(\mu_B) - f(\mu_A), f(1-\nu_A) - f(1-\nu_B) + 1) =$$
$$= (f'(\mu_C)(\mu_B-\mu_A), f'(1-\nu_C)(\nu_B-\nu_A)+1).$$

On the other hand

$$\bar{f}'(C)(B-A) = (f'(\mu_C), 1 - f'(1-\nu_C))(\mu_B-\mu_A, \nu_B-\nu_A+1) =$$

$$= (f'(\mu_C)(\mu_B-\mu_A), 1-(1-(1-f'(1-\nu_C)))(1-(\nu_B-\nu_A+1))) =$$
$$= (f'(\mu_C)(\mu_B-\mu_A), 1-f'(1-\nu_C)(\nu_A-\nu_B)) =$$
$$= (f'(\mu_C)(\mu_B-\mu_A), f'(1-\nu_C)(\nu_B-\nu_A)+1) = \bar{f}(B) - \bar{f}(A).$$

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