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INTUITIONISTIC FUZZY SETS

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A definition of the concept 'intuitionistic fuzzy set' (IFS) is given, the latter being a generalization of the concept 'fuzzy set' and an example is described. Various properties are proved, which are connected to the operations and relations over sets, and with modal and topological operators, defined over the set of IFS's.

Keywords: Intuitionistic fuzzy set, Fuzzy Set, Modal operator (necessity, possibility), Topological operator (interior, closure).

The notions of intuitionistic fuzzy set (IFS) and intuitionistic L-fuzzy sets (ILFS) were introduced in [1, 2] and [3], respectively, as a generalization of the notion of fuzzy set (FS).

Let a set E be fixed. An ILFS A^* in E is an object having the form

$$A^* = \{\langle x, \mu_A(x), \nu_A(x)\rangle \mid x \in E\},$$

where the functions $\mu_A : E \to L$ and $\nu_A : E \to L$ define the degree of membership and the degree of nonmembership of the element $x \in E$ to $A \subset E$ (for simplicity below we shall write A instead of A^*), respectively, the functions μ_A and ν_A should satisfy the condition:

$$(\forall x \in E)(\mu_A(x) \leqslant N(\nu_A(x))),$$

where $N : L \to L$ is an involutive order reversing operation in the lattice $\langle L, \leqslant\rangle$. When $L = [0, 1]$, the object A is an IFS and the following condition holds:

$$(\forall x \in E)(0 \leqslant \mu_A(x) + \nu_A(x) \leqslant 1).$$

Obviously every FS has the form $\{\langle x, \mu_A(x), 1 - \mu_A(x)\rangle \mid x \in E\}$.

Let us denote everywhere for simplicity by $\{\langle x, \mu_A(x)\rangle \mid x \in E\}$ the set $\{\langle x, \mu_A(x), 1 - \mu_A(x)\rangle \mid x \in E\}$. The definition makes clear that for the so constructed new type of FS the logical law of the excluded middle is not valid, similarly to the case in intuitionistic mathematics. Herefrom emerges the name of that set. Note that in the case when L is a lattice it is possible to introduce an example about a set for which $\mu_A(x)$, $N(\mu_A(x)) < \sup L$, where $\sup L$ is a maximal element of L, but as in that case $\nu_A(x) = N(\mu_A(x))$, the FS $\{\langle x, \mu_A(x)\rangle \mid x \in E\}$ cannot be equivalent to the IFS $\{\langle x, \mu_A(x), \nu_A(x)\rangle \mid x \in E\}$.

Before describing the properties of IFS's we shall give an example of an IFS which is not an FS. The example is written after an idea of the author in [4].

Let A, B, C and D be four convex, closed, connected and compact sets in the Euclidean plane, as shown in Figure 1 ($A \cap B = A \cap C = A \cap D = B \cap C = B \cap D = C \cap D = \emptyset$). Let in that plane Cartesian coordinates Ox_1x_2 be given and let the sets $P \cup Q \cup R$, $R \cup S \cup T$, $Q \cup R \cup S$ and U be respectively their orthogonal projections over the axis Ox_1. By the real number $l(y, X)$ we denote the length (as regards the length unit introduced for the axis Ox_1) of the segment in the set X lying on a line perpendicular to the axis Ox_1, passing through point y from Ox_1. Let $E = A \cup B \cup C \cup D$ be the universum for our further considerations, and let the sets F and G satisfy the following conditions:

(a) $A \subsetneqq F \subsetneqq A \cup C \cup D$;

(b) $B \subsetneqq G \subsetneqq B \cup C \cup D$;

(c) $F \cap G = \emptyset$;

(d) $F \cup G \subsetneqq E$.

From the last two conditions it follows that the set F is strictly included in the supplement of the set G to E; and from the first two – that the trivial cases $F = A$ and $G = B$ are excluded. The four conditions are independent.

Let us have the possibility to observe only the projections of the points from E over the axis Ox_1, and for every $x \in E$ we know only (observing its projection $y \in P \cup Q \cup R \cup S \cup T \cup U$) the value of $l(y, X)$, where X is some of the sets A, B, C and D.

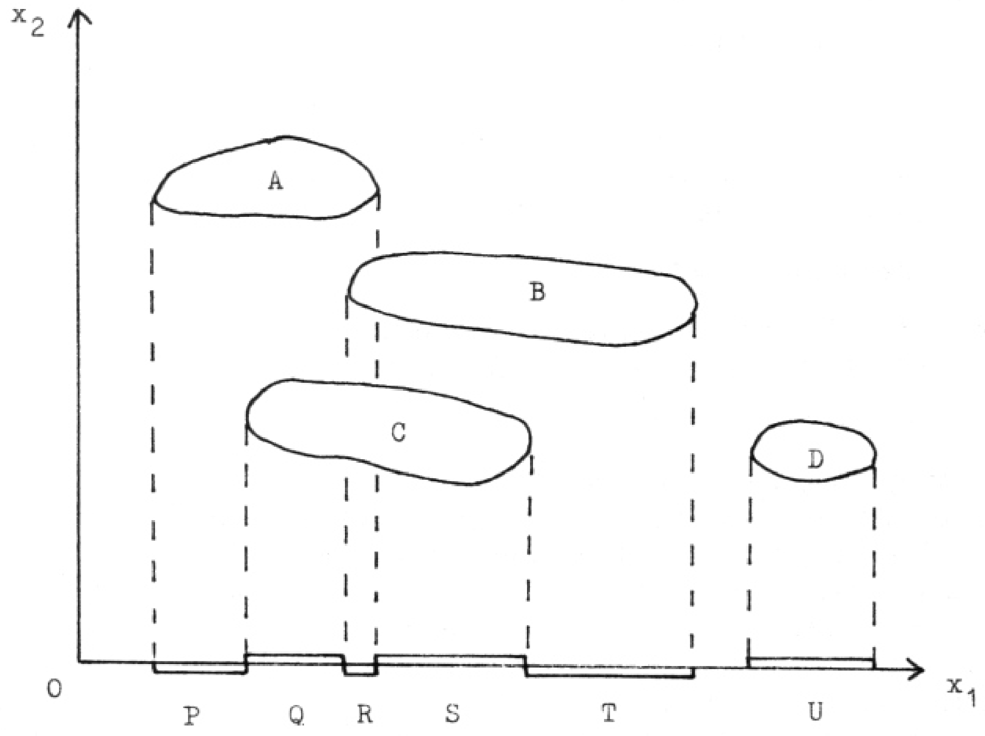

Fig. 1.

Our aim will be to show the form of the membership and non-membership functions of the elements of E towards the set F, with respect to the disposition of the four sets in the plane, as shown in Figure 1:

$$\mu_F(x) = \begin{cases} 1 & \text{if } y \in P, \\ \dfrac{l(y, A)}{l(y, A) + l(y, C)} & \text{if } y \in Q, \\ \dfrac{l(y, A)}{l(y, A) + l(y, B) + l(y, C)} & \text{if } y \in R, \\ 0 & \text{if } y \in S \cup T \cup U, \end{cases}$$

$$\nu_F(x) = \begin{cases} 0 & \text{if } y \in P \cup Q \cup U, \\ \dfrac{l(y, B)}{l(y, A) + l(y, B) + l(y, C)} & \text{if } y \in R, \\ \dfrac{l(y, B)}{l(y, B) + l(y, C)} & \text{if } y \in S, \\ 1 & \text{if } y \in T. \end{cases}$$

Then

$$\mu_F(x) + \nu_F(x) = \begin{cases} 1 & \text{if } y \in P \cup T, \\ \dfrac{l(y, A)}{l(y, A) + l(y, C)} & \text{if } y \in Q, \\ \dfrac{l(y, A) + l(y, B)}{l(y, A) + l(y, B) + l(y, C)} & \text{if } y \in R, \\ \dfrac{l(y, B)}{l(y, B) + l(y, C)} & \text{if } y \in S, \\ 0 & \text{if } y \in U, \end{cases}$$

i.e. $0 \leqslant \mu_F(x) + \nu_F(x) \leqslant 1$.

If $\pi_F(x) = 1 - \mu_F(x) - \nu_F(x)$, then $\pi_F(x)$ is the degree of indeterminacy of the element $x \in E$ to the set $F \subset E$. In the example we may write

$$\pi_F(x) = \begin{cases} 0 & \text{if } y \in P \cup T, \\ \dfrac{l(y, C)}{l(y, A) + l(y, C)} & \text{if } y \in Q, \\ \dfrac{l(y, C)}{l(y, A) + l(y, B) + l(y, C)} & \text{if } y \in R, \\ \dfrac{l(y, C)}{l(y, B) + l(y, C)} & \text{if } y \in S, \\ 1 & \text{if } y \in U. \end{cases}$$

Obviously, the form of the functions μ_F and ν_F may be simplified to

$$\mu_F(x)=\begin{cases}1 & \text{if } y\in P,\\ a & \text{if } y\in Q,\\ b & \text{if } y\in R,\\ c & \text{if } y\in S,\\ 0 & \text{if } y\in T,\\ d & \text{if } y\in U,\end{cases}$$

and

$$\nu_F(x)=\begin{cases}0 & \text{if } y\in P,\\ 1-a & \text{if } y\in Q,\\ 1-b & \text{if } y\in R,\\ 1-c & \text{if } y\in S,\\ 1 & \text{if } y\in T,\\ 1-d & \text{if } y\in U,\end{cases}$$

where $0\leqslant a, b, c, d\leqslant 1$ and then $\mu_F(x)+\nu_F(x)=1$, i.e. the set with these functions μ_F and ν_F is already a fuzzy one, but not yet a genuine IFS. However, as long as the elements of C and D are considered, it is not known which of them is from F and which from G, and which is not at all belonging to $F\cup G$; such a substitution for the values of μ_F and ν_F is not correct.

Above we noted the fact that we will consider the sets shown in Figure 1. The reason for this was that by the so selected configuration we may demonstrate easily the various possible values acquired by the functions μ_F and ν_F. In this way for example, the values of $x\in E$ and respectively $y\in P\cup Q\cup R\cup S\cup T\cup U$ are seen. For them it is absolutely sure that they belong (or not) to F (when $y\in P$, i.e. $x\in A$, and respectively $y\in T$, i.e. $x\in B$) or there is absolute indeterminacy (when $y\in U$, i.e. $x\in D$).

With the example constructed above we have demonstrated the existance of a genuine IFS, i.e. an IFS, which is not an FS.

For every two IFS's A and B the following operations and relations are valid:

(1) $A\subset B$ iff $(\forall x\in E)(\mu_A(x)\leqslant\mu_B(x)\ \&\ \nu_A(x)\geqslant\nu_B(x))$;

(2) $A=B$ iff $A\subset B\ \&\ B\subset A$;

(3) $\bar{A}=\{\langle x, \nu_A(x), \mu_A(x)\rangle \mid x\in E\}$;

(4) $A\cap B=\{\langle x, \min(\mu_A(x), \mu_B(x)), \max(\nu_A(x), \nu_B(x))\rangle \mid x\in E\}$;

(5) $A\cup B=\{\langle x, \max(\mu_A(x), \mu_B(x)), \min(\nu_A(x), \nu_B(x))\rangle \mid x\in E\}$;

(6) $A+B=\{\langle x, \mu_A(x)+\mu_B(x)-\mu_A(x)\cdot\mu_B(x), \nu_A(x)\cdot\nu_B(x)\rangle \mid x\in E\}$;

(7) $A\cdot B=\{\langle x, \mu_A(x)\cdot\mu_B(x), \nu_A(x)+\nu_B(x)-\nu_A(x)\cdot\nu_B(x)\rangle \mid x\in E\}$.

It is easy to verify the correctness of the defined operations and relations. For example, regarding the operation + (for the first three ones the check is obvious, and for operation $\cdot$ it is similar to that for $+$) it is enough to consider the four real numbers a, b, c, d ($\geqslant 0$), for which $0 \leqslant a+b \leqslant 1$, $0 \leqslant c+d \leqslant 1$. For them from $a(1-c)+c \geqslant c \geqslant 0$ it follows that

$$0 \leqslant b \cdot d \leqslant a+c-a \cdot c+b \cdot d \leqslant a+c-a \cdot c+(1-a)(1-c)=1.$$

From these inequations it follows directly that $A+B$ is an IFS.

Theorem 1 [1, 2]. (a) *The operations $\cap$ and $\cup$ are commutative, associative, distributive to the left and to the right among themselves, idempotent, and satisfy the law of De Morgan.*

(b) *The operations $+$ and $\cdot$ are commutative, associative, and satisfy a law, similar to the De Morgan law.*

(c) *The operations $+$ and $\cdot$ are distributive to the left and to the right with respect to the operations $\cap$ and $\cup$.*

We shall define over the set of all IFS's two operators which will transform every IFS into an FS, i.e. IFS. They are similar to the operators 'necessity' and 'possibility' defined in some modal logics. for every IFS A,

$$\Box A=\{\langle x, \mu_A(x)\rangle \mid x \in E\}=\{\langle x, \mu_A(x), 1-\mu_A(x)\rangle \mid x \in E\},$$
$$\Diamond A=\{\langle x, 1-\nu_A(x)\rangle \mid x \in E\}=\{\langle x, 1-\nu_A(x), \nu_A(x)\rangle \mid x \in E\}.$$

Theorem 2 [1, 2]. *For every IFS A:*

(a) $\Box A=\overline{\Diamond \bar{A}}$;
(b) $\Diamond A=\overline{\Box \bar{A}}$;
(c) $\Box A \subset A \subset \Diamond A$;
(d) $\Box\Box A=\Box A$;
(e) $\Box\Diamond A=\Diamond A$;
(f) $\Diamond\Box A=\Box A$;
(g) $\Diamond\Diamond A=\Diamond A$.

Theorem 3 [1, 2]. *For every two IFS's A and B:*

(a) $\Box(A \cup B)=\Box A \cup \Box B$;
(b) $\Diamond(A \cup B)=\Diamond A \cup \Diamond B$.

Here we shall prove the following:

Theorem 4. *For every two IFS's A and B:*

(a) $\Box(A \cap B)=\Box A \cap \Box B$;
(b) $\Diamond(A \cap B)=\Diamond A \cap \Diamond B$;
(c) $\Box(A+B)=\Box A+\Box B$;
(d) $\Diamond(A+B)=\overline{\Diamond A \cdot \Diamond B}$;
(e) $\Box(A \cdot B)=\overline{\Box A \cdot \Box B}$;
(f) $\Diamond(A \cdot B)=\overline{\Diamond A+\Diamond B}$.

Proof. (a):

$$\begin{aligned}\Box(A\cap B)&=\Box\{\langle x,\min(\mu_A(x),\mu_B(x)),\max(\nu_A(x),\nu_B(x))\rangle\mid x\in E\}\\&=\{\langle x,\min(\mu_A(x),\mu_B(x))\rangle\mid x\in E\}\\&=\{\langle x,\mu_A(x)\rangle\mid x\in E\}\cap\{\langle x,\mu_B(x)\rangle\mid x\in E\}\\&=\Box A\cap\Box B;\end{aligned}$$

(f):

$$\begin{aligned}\Diamond(A\cdot B)&=\Diamond\{\langle x,\mu_A(x)\cdot\mu_B(x),\nu_A(x)+\nu_B(x)-\nu_A(x)\cdot\nu_B(x)\rangle\mid x\in E\}\\&=\{\langle x,1-\nu_A(x)-\nu_B(x)+\nu_A(x)\cdot\nu_B(x)\rangle\mid x\in E\}\\&=\overline{\{\langle x,\nu_A(x)+\nu_B(x)-\nu_A(x)\cdot\nu_B(x)\rangle\mid x\in E\}}\\&=\overline{\{\langle x,\nu_A(x)\rangle\mid x\in E\}+\{\langle x,\nu_B(x)\rangle\mid x\in E\}}\\&=\overline{\Diamond A+\Diamond B}.\end{aligned}$$

(b)–(e) are proved analogously.

The following two relations are possible [4]:

$$A\subset_\Box B\quad\text{iff}\quad(\forall x\in E)(\mu_A(x)\leqslant\mu_B(x)),$$
$$A\subset_\Diamond B\quad\text{iff}\quad(\forall x\in E)(\nu_A(x)\geqslant\nu_B(x)).$$

A justification for the notions $\subset_\Box$ and $\subset_\Diamond$ is the following:

Theorem 5. *For every two IFS's A and B:*
(a) $A\subset_\Box B$ *iff* $\Box A\subset\Box B$;
(b) $A\subset_\Diamond B$ *iff* $\Diamond A\subset\Diamond B$.

Proof. (a) Let $A\subset_\Box B$, i.e. $(\forall x\in E)(\mu_A(x)\leqslant\mu_B(x))$. Then for $\Box A$ and $\Box B$ it follows that $\Box A\subset\Box B$. Contrary, if $\Box A\subset\Box B$, then $(\forall x\in E)(\mu_A(x)\leqslant\mu_B(x))$, i.e. $A\subset_\Box B$.
(b) is proved analogously.

We shall introduce one more relation:

$$A\sqsubset B\quad\text{iff}\quad(\forall x\in E)(\pi_A(x)\leqslant\pi_B(x)).$$

Theorem 6. *For every two IFS's A and B:*
(a) $A\subset_\Box B\ \&\ A\subset_\Diamond B$ *iff* $A\subset B$;
(b) *if* $A\subset_\Box B\ \&\ A\sqsubset B$, *then* $A\subset B$;
(c) *if* $A\subset_\Diamond B\ \&\ B\sqsubset A$, *then* $A\subset B$.

Proof. (a) If $A\subset_\Box B\ \&\ A\subset_\Diamond B$ then

$$(\forall x\in E)(\mu_A(x)\leqslant\mu_A(x))\ \&\ (\forall x\in E)(\nu_A(x)\geqslant\nu_B(x)).$$

Hence $(\forall x\in E)(\mu_A(x)\leqslant\mu_B(x)\ \&\ \nu_A(x)\geqslant\nu_B(x))$. Therefore $A\subset B$. Contrary, if $A\subset B$, then

$$(\forall x\in E)(\mu_A(x)\leqslant\mu_B(x)\ \&\ \nu_A(x)\geqslant\nu_B(x)),$$

i.e. $A\subset_\Box B\ \&\ A\subset_\Diamond B$.

(b) and (c) are proved analogously.

Let A be a given IFS. We determine for its the four numbers

$$K=\max_{x\in E}\mu_A(x),\qquad L=\min_{x\in E}\nu_A(x),$$

$$k=\min_{x\in E}\mu_A(x),\qquad l=\max_{x\in E}\nu_A(x)$$

and the sets

$$C(A)=\{\langle x,K,L\rangle\mid x\in E\},\qquad I(A)=\{\langle x,k,l\rangle\mid x\in E\}$$

which will be called closure and interior.

Obviously $0\leqslant K+L\leqslant 1$, because if $K=\mu_A(x_1)$ for some $x_1\in E$ then $0\leqslant \mu_A(x_1)+\nu_A(x_1)\leqslant 1$, but if $L\leqslant\nu_A(x_1)$, then $x_2\in E$ will exists such that $L=\nu_A(x_2)\leqslant\nu_A(x_1)$ and therefore $0\leqslant K+L\leqslant 1$. Analogously $0\leqslant k+l\leqslant 1$. Herefrom it follows directly that the following is valid:

Theorem 7. *For every two IFS's A and B:*

(a) *$C(A)$ and $I(A)$ are IFS's;*
(b) *$I(A)\subset A\subset C(A)$;*
(c) *$C(A\cup B)=C(A)\cup C(B)$;*
(d) *$C(C(A))=C(A)$;*
(e) *$C(\bar{0})=\bar{0}$, where $\bar{0}=\{\langle x,0,1\rangle\mid x\in E\}$.*

Proof. The validity of (a), (b) and (e) is obvious.

(c):

$$\begin{aligned}C(A\cup B)&=C(\{\langle x,\max(\mu_A(x),\mu_B(x)),\min(\nu_A(x),\nu_B(x))\rangle\mid x\in E\})\\&=\{\langle x,K,L\rangle\mid x\in E\},\end{aligned}$$

where

$$K=\max_{x\in E}(\max(\mu_A(x),\mu_B(x)))=\max\left(\max_{x\in E}\mu_A(x),\max_{x\in E}\mu_B(x)\right),$$

$$L=\min_{x\in E}(\min(\nu_A(x),\nu_B(x)))=\min\left(\min_{x\in E}\nu_A(x),\min_{x\in E}\nu_B(x)\right).$$

Then

$$\begin{aligned}C(A\cup B)&=\left\{\left\langle x,\max\left(\max_{x\in E}\mu_A(x),\max_{x\in E}\mu_B(x)\right),\min\left(\min_{x\in E}\nu_A(x),\min_{x\in E}\nu_B(x)\right)\right\rangle\,\middle|\,x\in E\right\}\\&=\left\{\left\langle x,\max_{x\in E}\mu_A(x),\min_{x\in E}\nu_A(x)\right\rangle\,\middle|\,x\in E\right\}\\&\qquad\cup\left\{\left\langle x,\max_{x\in E}\mu_B(x),\min_{x\in E}\nu_B(x)\right\rangle\,\middle|\,x\in E\right\}\\&=C(A)\cup C(B);\end{aligned}$$

(d):

$$\begin{aligned}C(C(A))&=C\left(\left\{\left\langle x,\max_{x\in E}\mu_A(x),\min_{x\in E}\nu_A(x)\right\rangle\,\middle|\,x\in E\right\}\right)\\&=\left\{\left\langle x,\max_{x\in E}\mu_A(x),\min_{x\in E}\nu_A(x)\right\rangle\,\middle|\,x\in E\right\}=C(A).\end{aligned}$$

From the validity of (c), (d), the right inclusion of (b) and (e), follows the validity of:

Theorem 8. *$\langle E, U, C\rangle$ is a topological space.*

Theorem 9. *For every IFS A:*

(a) $\Diamond A \subset C(A)$;

(b) $I(A) \subset \Box A$;

(c) $\overline{I(\bar{A})} = C(A)$.

Proof. (a) $C(A) = \{\langle x, K, L\rangle \mid x \in E\}$, where K and L have the above form. Because

$$(\forall x \in E)(K \geqslant 1 - \mu_A(x) \ \&\ L \leqslant \nu_A(x)),$$

from the definition of the relation $\subset$ it follows that $\Diamond A \subset C(A)$;

(b) is proved analogously;

(c):

$$\begin{aligned} I(\bar{A}) &= \overline{I(\{\langle x, \nu_A(x), \mu_A(x)\rangle \mid x \in E\})} \\ &= \overline{\{\langle x, L, K\rangle \mid x \in E\}} \\ &= \{\langle x, K, L\rangle \mid x \in E\} = C(A). \end{aligned}$$

Theorem 10. *For every IFS A:*

(a) $\Box C(\Box A) \subset C(A)$, $\Diamond C(\Box A) \subset C(A)$, $\overline{\Box I(\Diamond \bar{A})} \subset C(A)$, $\overline{\Diamond I(\Diamond \bar{A})} \subset C(A)$;

(b) $\Box C(\Diamond A) \supset C(A)$, $\Diamond C(\Diamond A) \supset C(A)$, $\overline{\Box I(\Box \bar{A})} \supset C(A)$, $\overline{\Diamond I(\Box \bar{A})} \supset C(A)$;

(c) $\Box I(\Box A) \subset I(A)$, $\Diamond I(\Box A) \subset I(A)$, $\overline{\Box C(\Diamond \bar{A})} \subset I(A)$, $\overline{\Diamond C(\Diamond \bar{A})} \subset I(A)$;

(d) $\Box I(\Diamond A) \supset I(A)$, $\Diamond I(\Diamond A) \supset I(A)$, $\overline{\Box C(\Box \bar{A})} \supset I(A)$, $\overline{\Diamond C(\Box \bar{A})} \supset I(A)$;

(e) $\Box C(\Box \bar{A}) \subset \overline{I(A)}$, $\Diamond C(\Box \bar{A}) \subset \overline{I(A)}$, $\overline{\Box I(\Diamond A)} \subset \overline{I(A)}$, $\overline{\Diamond I(\Diamond A)} \subset \overline{I(A)}$;

(f) $\Box C(\Diamond \bar{A}) \supset \overline{I(A)}$, $\Diamond C(\Diamond \bar{A}) \supset \overline{I(A)}$, $\overline{\Box I(\Box A)} \supset \overline{I(A)}$, $\overline{\Diamond I(\Box A)} \supset \overline{I(A)}$;

(g) $\Box I(\Box \bar{A}) \subset \overline{C(A)}$, $\Diamond I(\Box \bar{A}) \supset \overline{C(A)}$, $\overline{\Box C(\Diamond A)} \subset \overline{C(A)}$, $\overline{\Diamond C(\Diamond A)} \supset \overline{C(A)}$;

(h) $\Box I(\Diamond \bar{A}) \supset \overline{C(A)}$, $\Diamond I(\Diamond \bar{A}) \supset \overline{C(A)}$, $\overline{\Box C(\Box A)} \supset \overline{C(A)}$, $\overline{\Diamond C(\Box A)} \supset \overline{C(A)}$.

Proof. (a):

$$\begin{aligned} \Box C(\Box A) &= \Box C(\Box \{\langle x, \mu_A(x), \nu_A(x)\rangle \mid x \in E\}) \\ &= \Box C(\{\langle x, \mu_A(x), 1 - \mu_A(x)\rangle \mid x \in E\}) \\ &= \Box \left\{\left\langle x, \max_{x \in E} \mu_A(x), \min_{x \in E} (1 - \mu_A(x)\right\}\right\rangle \,\middle|\, x \in E\right\} \\ &= \left\{\left\langle x, \max_{x \in E} \mu_A(x), 1 - \max_{x \in E} \mu_A(x)\right\rangle \,\middle|\, x \in E\right\} \\ &= \{\langle x, K, 1 - K\rangle \mid x \in E\} \\ &\subset \{\langle x, K, L\rangle \mid x \in E\} = C(A); \end{aligned}$$

$\diamond C(\square A) \subset C(A)$ is proved analogously;

$$\begin{aligned}
\overline{\square I(\diamond \bar{A})} &= \overline{\square I(\diamond\{\langle x, \nu_A(x), \mu_A(x)\rangle \mid x \in E\})} \\
&= \overline{\square I(\{\langle x, 1-\mu_A(x), \mu_A(x)\rangle \mid x \in E\})} \\
&= \overline{\square\left\{\left\langle x, \min_{x\in E}(1-\mu_A(x)), \max_{x\in E}\mu_A(x)\right\rangle \,\middle|\, x \in E\right\}} \\
&= \overline{\left\{\left\langle x, \min_{x\in E}(1-\mu_A(x)), 1-\min_{x\in E}(1-\mu_A(x))\right\rangle \,\middle|\, x \in E\right\}} \\
&= \left\{\left\langle x, 1-\min_{x\in E}(1-\mu_A(x)), \min_{x\in E}(1-\mu_A(x))\right\rangle \,\middle|\, x \in E\right\} \\
&= \left\{\left\langle x, \max_{x\in E}\mu_A(x), 1-\max_{x\in E}\mu_A(x)\right\rangle \,\middle|\, x \in E\right\} \\
&= \{\langle x, K, 1-K\rangle \mid x \in E\} \\
&\subset \{\langle x, K, L\rangle \mid x \in E\} = C(A).
\end{aligned}$$

All the other assertions are proved in the same way.

Finally we shall define one more operation over IFS's – the Cartesian product.

Let E_1 and E_2 be two universums and let $A = \{\langle x_1, \mu_A(x_1), \nu_A(x_1) \mid x_1 \in E_1\}$ and $B = \{\langle x_2, \mu_B(x_2), \nu_B(x_2)\rangle \mid x_2 \in E_2\}$ be two IFS's. The Cartesian product of these two IFS's we shall call the set

$$A \times B = \{\langle\langle x_1, x_2\rangle, \mu_A(x_1)\cdot\mu_B(x_2), \nu_A(x_1)\cdot\nu_B(x_2)\rangle \mid x_1 \in E_1, x_2 \in E_2\}.$$

Because $0 \leqslant \mu_A(x_1)\cdot\mu_B(x_2) + \nu_A(x_1)\cdot\nu_B(x_2) \leqslant \mu_A(x_1) + \nu_A(x_1) \leqslant 1$, it follows that $A \times B$ is an IFS, but now with universum $E_1 \times E_2$.

Theorem 11. *For every three universums E_1, E_2, E_3 and four IFS's A, B (over E_1), C (over E_2), D (over E_3):*

(a) $A \times C = C \times A$;
(b) $(A \times C) \times D = A \times (C \times D)$;
(c) $(A \cup B) \times C = (A \times C) \cup (B \times C)$;
(d) $(A \cap B) \times C = (A \times C) \cap (B \times C)$;
(e) $(A + B) \times C \subset (A \times C) + (B \times C)$;
(f) $(A \cdot B) \times C \supset (A \times C)\cdot(B \times C)$.

Proof. The validity of the first four assertions follows from the definitions. We shall prove (e): First,

$$\begin{aligned}
(A+B)\times C &= \{\langle x, \mu_A(x)+\mu_B(x)-\mu_A(x)\cdot\mu_B(x), \\
&\qquad \nu_A(x)\cdot\nu_B(x)\rangle \mid x \in E\} \times C \\
&= \{\langle\langle x, y\rangle, (\mu_A(x)+\mu_B(x)-\mu_A(x)\cdot\mu_B(x))\cdot\mu_C(y), \\
&\qquad \nu_A(x)\cdot\nu_B(x)\cdot\nu_C(y)\rangle \mid x \in E_1, y \in E_2\}.
\end{aligned}$$

However,

$$\begin{aligned}
&\mu_A(x)\cdot\mu_C(y)+\mu_B(x)\cdot\mu_C(y)-\mu_A(x)\cdot\mu_B(x)\cdot\mu_C(y) \\
&\quad \leqslant \mu_A(x)\cdot\mu_C(y)+\mu_B(x)\cdot\mu_C(y)-\mu_A(x)\cdot\mu_B(x)\cdot\mu_C(y)^2
\end{aligned}$$

and

$$\nu_A(x)\cdot\nu_B(x)\cdot\nu_C(y)\geqslant\nu_A(x)\cdot\nu_B(x)\cdot\nu_C(y)^2,$$

so that

$$\begin{aligned}(A+B)\times C\subset\{\langle\langle x,y\rangle,\ &\mu_A(x)\cdot\mu_C(y)+\mu_B(x)\cdot\mu_C(y)-\mu_A(x)\cdot\mu_B(x)\cdot\mu_C(y)^2,\\ &\times\nu_A(x)\cdot\nu_B(x)\cdot\nu_C(y)^2\rangle\mid x\in E_1,\ y\in E_2\}\\ &=\{\langle\langle x,y\rangle,\mu_A(x)\cdot\mu_C(y),\nu_A(x)\cdot\nu_C(y)\rangle\mid x\in E_1,\ y\in E_2\}\\ &\quad+\{\langle\langle x,y\rangle,\mu_B(x)\cdot\mu_C(y),\nu_B(x)\cdot\nu_C(y)\rangle\mid x\in E_1,\ y\in E_2\}\\ &=(A\times C)+(B\times C).\end{aligned}$$

Similarly also (f) can be proved.

Theorem 12. *For every two universums E_1, E_2 and two IFS's A, B over them:*

(a) $\Box(A\times B)\subset\Box A\times\Box B$;

(b) $\Diamond(A\times B)\supset\Diamond A\times\Diamond B$;

(c) $C(A\times B)\subset C(A)\times C(B)$;

(d) $I(A\times B)\supset I(A)\times I(B)$.

Proof. (a):

$$\begin{aligned}\Box(A\times B)&=\Box\{\langle\langle x,y\rangle,\mu_A(x)\cdot\mu_B(y),\nu_A(x)\cdot\nu_B(y)\rangle\mid x\in E_1,\ y\in E_2\}\\ &=\{\langle\langle x,y\rangle,\mu_A(x)\cdot\mu_B(y),1-\mu_B(x)\cdot\mu_B(t)\rangle\mid x\in E_1,\ y\in E_2\}\\ &\qquad\text{(from the obvious inequation } a+b\geqslant 2ab,\text{ for } 0\leqslant a,b\leqslant 1)\\ &\subset\{\langle\langle x,y\rangle,\mu_A(x)\cdot\mu_B(y),\\ &\qquad\qquad(1-\mu_A(x))\cdot(1-\mu_B(y))\rangle\mid x\in E_1,\ y\in E_2\}\\ &=\{\langle x,\mu_A(x),1-\mu_A(x)\rangle\mid x\in E_1\}\times\{\langle y,\mu_B(y),1-\mu_B(y)\rangle\mid y\in E_2\}\\ &=\Box A\times\Box B.\end{aligned}$$

The other assertions can be checked in a similar way.

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References

[1] K. Atanassov, Intuitionistic fuzzy sets, in: V. Sgurev, Ed., VII ITKR's Session, Sofia, June 1983 (Central Sci. and Techn. Library, Bulg. Academy of Sciences, 1984).

[2] K. Atanassov and S. Stoeva, Intuitionistic fuzzy sets, in: Polish Symp. on Interval & Fuzzy Mathematics, Poznan (Aug. 1983) 23–26.

[3] K. Atanassov and S. Stoeva, Intuitionstic L-fuzzy sets, in: R. Trappl, Ed., Cybernetics and Systems Research 2 (Elsevier Sci. Publ., Amsterdam, 1984) 539–540.

[4] K. Atanassov, Intuitionistic fuzzy relations, in: L. Antonov, Ed., III International School "Automation and Scientific Instrumentation", Varna (Oct. 1984) 56–57.